

THE FARRELL-JONES CONJECTURE FOR MAPPING CLASS GROUPS

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ABSTRACT. We prove the Farrell-Jones Conjecture for mapping class groups. The proof uses the Masur-Minsky theory of the large scale geometry of mapping class groups and the geometry of the thick part of Teichmüller space. The proof is presented in an axiomatic setup, extending the projection axioms in [12]. More specifically, we prove that the action of $\text{Mod}(\Sigma)$ on the Thurston compactification of Teichmüller space is finitely \mathcal{F} -amenable for the family \mathcal{F} consisting of virtual point stabilizers.

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INTRODUCTION

The goal of this paper is to prove the following theorem.

Theorem A. *The mapping class group $\text{Mod}(\Sigma)$ of any oriented surface Σ of finite type satisfies the Farrell-Jones Conjecture.*

We will review the Farrell-Jones Conjecture and its motivation later in the introduction. The main step in our proof of Theorem A is the verification of a regularity condition, called finite \mathcal{F} -amenability in [2], for the action of $\text{Mod}(\Sigma)$ on the space \mathcal{PMF} of projective measured foliations on Σ , see Theorem B below. Theorem A is then a consequence of the axiomatic results of Lück, Reich and the first author for the Farrell-Jones Conjecture from [5, 7] and an induction on the complexity of the surface. A similar induction has been used for $\text{GL}_n(\mathbb{Z})$ [9].

Topologically amenable actions. Let G be a group. The space $\text{Prob}(G)$ of probability measures on G is a subspace of $\ell^1(G)$. An action of G on a compact space Δ is said to be *topologically amenable* if there exists a sequence of weak* continuous maps $f_n: \Delta \rightarrow \text{Prob}(G)$ satisfying the

following condition: for any $g \in G$

$$\sup_{x \in \Delta} \|f_n(gx) - gf_n(x)\|_1 \rightarrow 0 \text{ as } n \rightarrow \infty.$$

It will be convenient to refer to sequences of maps satisfying this condition as *almost equivariant maps*. Groups that admit a topologically amenable action on a compact space are often said to be *boundary amenable*. This condition is known to imply the Novikov conjecture [35]. Since the Stone-Ćech compactification βG maps to any compact G -space, it follows that a group is boundary amenable if and only if its action on βG is topologically amenable. Hamenstädt proved that the mapping class group of a surface Σ is boundary amenable by proving that its action on the space of complete geodesic laminations on Σ is topologically amenable [32]. In particular, the action of the mapping class group on its Stone-Ćech compactification is topologically amenable.

Finite asymptotic dimension. For any N the subspace $\text{Prob}(G)^{(N)}$ of probability measures supported on sets of cardinality at most $N + 1$ is naturally a simplicial complex of dimension N . The isotropy groups for this action all belong to the family Fin of finite subgroups. For an amenable action of G on a compact space Δ one can ask whether the almost equivariant maps f_n from above can be chosen to have image in $\text{Prob}(G)^{(N)}$ for some N independent of n . For $\Delta = \beta G$ such maps f_n exist if and only if the asymptotic dimension of G is $\leq N$ [30, Thm. 6.5]. Together with Bromberg and Fujiwara the second author proved that the mapping class group has finite asymptotic dimension [12]. This result implies a stronger form of the Novikov conjecture [42], often called the integral Novikov conjecture, i.e., the integral injectivity of the assembly maps of the assembly maps in algebraic K -theory and L -theory relative to the family of finite subgroups. These assembly maps are briefly reviewed in Subsection 4.B.

Finite \mathcal{F} -amenability. The axiomatic condition from [7] that we will use requires actions on spaces much nicer than the Stone-Ćech compactification. For the mapping class group this will be the Thurston compactification of Teichmüller space, i.e., a disk. Technically, the requirement is that the space is a Euclidean retract. For actions on such spaces there are typically infinite isotropy groups and this obstructs the existence of almost equivariant maps f_n into a finite dimensional simplicial complex with a proper simplicial G -action. The condition from [7] is therefore formulated relative to a family \mathcal{F} of subgroups; the action on the target simplicial complex is then allowed to have

isotropy in this family. Maps to a finite dimensional simplicial space translate to finite dimensional covers of the source, as we can pull back standard coverings of the simplicial complex. This translation is used in the formulation the regularity condition that we recall now in detail.

Let \mathcal{F} be a *family of subgroups* of a group G , i.e., \mathcal{F} is set of subgroups of G that is closed under conjugation and taking subgroups. A subset U of a G -space is said to be an \mathcal{F} -*subset* if there is $F \in \mathcal{F}$ such that $gU = U$ for all $g \in F$ and $gU \cap U = \emptyset$ if $g \notin F$. A cover \mathcal{U} of a G -space is said to be an \mathcal{F} -*cover* if \mathcal{U} is G -invariant and consists of \mathcal{F} -subsets. If all members of \mathcal{U} are in addition open, then we say that \mathcal{U} is an open \mathcal{F} -cover. For $N \in \mathbb{N}$ an action of G on a space Δ is said to be N - \mathcal{F} -*amenable* if for all finite subsets S of G there exists an open \mathcal{F} -cover \mathcal{U} of $G \times \Delta$ (equipped with the diagonal G -action $g \cdot (h, x) = (gh, gx)$) such that

- the order of \mathcal{U} is at most N ;
- the cover \mathcal{U} is S -*long* (in the group coordinate), i.e., for every $(g, x) \in G \times \Delta$ there is $U \in \mathcal{U}$ with $gS \times \{x\} \subseteq U$.

An action that is N - \mathcal{F} -amenable for some N is said to be *finitely \mathcal{F} -amenable*.

Given such a cover one obtains almost equivariant maps f_n from Δ to simplicial complexes of dimension $\leq N$ as follows. For each \mathcal{U} a partition of unity subordinate to \mathcal{U} provides a G -equivariant map $f_{\mathcal{U}}$ from $G \times \Delta$ to the Nerve of \mathcal{U} . By the first condition this nerve is of dimension at most N . The second condition translates into the almost equivariance of the restrictions of the $f_{\mathcal{U}}$ to $\{e\} \times \Delta$.

It is straightforward to check that the action of the mapping class group on Teichmüller space is finitely Fin -amenable. The key point for us is to understand the action on the boundary of Teichmüller space, i.e., on the space \mathcal{PMF} of projective measured foliations.

Theorem B. *Let Σ be a closed oriented surface of genus g with p punctures where $6g + 2p - 6 > 0$. Let \mathcal{F} be the family of subgroups that virtually fix an essential simple closed curve (up to isotopy) or are virtually cyclic. Then the action of its mapping class group $\text{Mod}(\Sigma)$ on the space \mathcal{PMF} of projective measured foliations on Σ is finitely \mathcal{F} -amenable.*

The family \mathcal{F} appearing in Theorem B can alternatively be described as the family of subgroups that virtually fix a point in \mathcal{PMF} : Firstly, every curve determines a point in \mathcal{PMF} . Secondly, every cyclic subgroup fixes a point in the Thurston compactification \mathcal{T} of Teichmüller space by the Brouwer fixed point theorem; since the action on Teichmüller space is proper, every infinite cyclic subgroup fixes a point

in \mathcal{PMF} . On the other hand, for any finitely \mathcal{F} -amenable action on any space all isotropy groups of the action necessarily belong to \mathcal{F} . Thus the family \mathcal{F} above is, up to finite index subgroups, the smallest family for which the action of the mapping class group on \mathcal{PMF} can be finitely \mathcal{F} -amenable.

Surface groups. To motivate our proof of Theorem B we recall the model argument, due to Farrell-Jones [24]. We later also discuss the situation of $\mathrm{SL}_2(\mathbb{Z})$, where the action is not cocompact. Let Σ be a closed hyperbolic surface. Thus $G = \pi_1(\Sigma)$ acts on the universal cover $\tilde{\Sigma} = \mathbb{H}^2$ and on the circle at infinity Δ . We sketch a proof that the action of G on Δ is finitely \mathcal{F} -amenable for the family \mathcal{F} consisting of cyclic subgroups. There are two steps in the proof, *long thin covers* and a *geodesic flow argument*.

Let $M = T_1\Sigma$ be the unit tangent bundle of Σ . This is a closed 3-manifold equipped with a 1-dimensional foliation by the orbits of the geodesic flow. Thus $v, w \in T_1\Sigma$ are in the same leaf if and only if there is a geodesic line in Σ tangent to both v and w . For any R there are only finitely many closed leaves of length $\leq R$.

Step 1: Long thin covers. For every $\epsilon > 0$ and $R > 0$ construct an open cover \mathcal{U} of M such that:

- every leaf segment of length $\leq 2R$ is contained in some $U \in \mathcal{U}$,
- every $U \in \mathcal{U}$ is contained in the ϵ -neighborhood of some leaf segment of length $\leq 9R$,
- the multiplicity of the cover is bounded above independently of R and ϵ .

The elements of the cover are going to be small neighborhoods of the closed leaves of length $\leq R$, and otherwise they will be flow boxes for the foliation of the form (leaf segment) \times (small cross section). Care must be taken to arrange the third bullet.

Lifting this cover to $T_1\mathbb{H}^2$ produces an open cover $\tilde{\mathcal{U}}$. This is going to be an \mathcal{F} -cover if ϵ is sufficiently small so that $\pi_1(U) \rightarrow \pi_1(M)$ has cyclic image for every U (nontrivial for neighborhoods of closed leaves, and otherwise trivial). The number R will depend on the given finite set $S \subset G$ and then ϵ depends on R .

Step 2: Geodesic flow argument. First define the *flow space*

$$FS = \{(x, p, \xi) \in \overline{\mathbb{H}^2} \times \mathbb{H}^2 \times \Delta \mid p \in [x, \xi]\}$$

where $[x, \xi]$ is the geodesic joining x to ξ . There is an embedding $T_1\mathbb{H}^2 \rightarrow FS$ onto a closed subset defined by $v \mapsto (\xi_-, p, \xi_+)$, where $p \in \mathbb{H}^2$ is the point where v is based, and $\xi_{\pm} \in \Delta$ are the points at $\pm\infty$ of the geodesic tangent to v . The construction of the \mathcal{F} -cover $\tilde{\mathcal{U}}$

in the first step can also be applied to FS . (Alternatively, it suffices to extend the cover of $T_1\mathbb{H}^2$ such that it covers a neighborhood of $T_1\mathbb{H}^2$ in FS preserving the multiplicity, see Lemma 3.16). Consider for any $\tau \geq 0$ the map

$$\iota_\tau: G \times \Delta \rightarrow FS$$

defined as follows. First identify G with an orbit in \mathbb{H}^2 , thus $G \subset \mathbb{H}^2$. Then let

$$\iota_\tau(g, \xi) = (g, x_\tau, \xi)$$

where x_τ is the unique point on the geodesic ray $[g, \xi)$ at distance τ from g . Thus ι_τ is the map “flow for time τ ”. Now we argue that for τ sufficiently large the cover $\iota_\tau^{-1}(\tilde{\mathcal{U}})$ satisfies the requirements. This is accomplished by a geometric limit argument: if the statement is false then for every τ we have (g_τ, ξ_τ) so that $\iota_\tau(B_R(g_\tau) \times \{\xi_\tau\})$ is not contained in any \tilde{U} . By equivariance we may assume that the middle coordinate of $\iota_\tau(g_\tau)$ belongs to a fixed compact set K (this uses cocompactness of the action) and passing to the limit as $\tau \rightarrow \infty$ produces $(\xi_-, p, \xi_+) \in T_1\mathbb{H}^2$ not contained in any \tilde{U} , contradiction. The key point here is that an R -ball gets squeezed by the geodesic flow to a long thin set.

The group $SL_2(\mathbb{Z})$. Let G be a torsion-free subgroup of finite index of $SL_2(\mathbb{Z})$. Here we sketch a proof (close to what we do for mapping class groups) that the standard action of G on the circle at infinity Δ is finitely \mathcal{F} -amenable for the family \mathcal{F} of cyclic subgroups. The proof for surface groups breaks down since the action on \mathbb{H}^2 is not cocompact and it is not possible to arrange that the point $\iota_\tau(g)$ belongs to a fixed compact set. Fix an equivariant, pairwise disjoint collection of horoballs, so that the action on the complement is cocompact and identify G with an orbit outside the horoballs. For a fixed $\Theta > 0$ say that the pair $(g, \xi) \in G \times \Delta$ is Θ -thick if the geodesic ray $[g, \xi)$ intersects every horoball in a segment of length $\leq \Theta$. Then the above argument generalizes to show that for any $\Theta > 0$ and any $S \subset G$ there is a required cover of the Θ -thick part of $G \times \Delta$. But we are still left to cover the thin part.

The naive idea is as follows. For every (g, ξ) which is not Θ -thick let $B(g, \xi)$ be the first horoball that the ray $[g, \xi)$ intersects in a segment of length $> \Theta$. Then for each horoball B define the set

$$U(B) = \{(g, \xi) \mid B(g, \xi) = B\}$$

Clearly, the collection $\{U(B)\}$ is equivariant, covers the Θ -thin part of $G \times \Delta$, consists of \mathcal{F} -sets, and any two are disjoint, i.e. the multiplicity is 1. The problem is that the sets $U(B)$ are not necessarily

open, and the cover may not be S -long. In both cases, the issue is the *threshold problem*, i.e., small perturbations of (g, ξ) may change $B(g, \xi)$ dramatically.

There are two things we do to solve the threshold problem. The first is to modify how we measure the size of intersection between a ray and a horoball. This is formalized in the notion of an *angle* and its main feature is that if a ray has long intersections with three horoballs, perturbing (g, ξ) does not affect the angle at the middle horoball (see Section 2.B). To define the angle we use the language of projection complexes [12], but in the present case one could use the Farey graph, whose vertices correspond to the horoballs. This leaves the threshold problem only in the case of the first large intersection, and we solve this by working at two (or technically six) different scales, see Section 2.C.

There is an alternative argument for $\mathrm{SL}_2(\mathbb{Z})$ where the failure of cocompactness is addressed on the flow space [9]. This alternative argument seems not to be applicable to the mapping class group, for example because we have only control over the behavior of Teichmüller rays that stay in a thick part.

Sketch of proof of Theorem B. Our argument follows the model arguments sketched above, but with several important differences. First, our geodesic flow argument, modelled on [2], is *coarse*, i.e. it squeezes balls to sets with bounded, rather than ϵ -small, cross-section. The flow space FS is replaced by the coarse flow space CF .

Second, instead of measuring lengths of intersections with horoballs, we use the Masur-Minsky notion of subsurface projection.

The hyperbolic plane is replaced with the Teichmüller space \mathcal{T} of complete hyperbolic structures of finite area on $\Sigma \setminus P$. It is equipped with the Teichmüller metric, which is invariant under the action of $\mathrm{Mod}(\Sigma)$, and it is compactified by \mathcal{PMF} . Fix a basepoint $X_0 \in \mathcal{T}$ and identify $\mathrm{Mod}(\Sigma)$ with the orbit of X_0 . Given a pair $(g, \xi) \in \mathrm{Mod}(\Sigma) \times \mathcal{PMF}$ there is a unique Teichmüller ray $c_{g, \xi}$ that starts at $g(X_0)$ and is “pointing towards ξ ” (technically, the vertical foliation of the quadratic differential is ξ). The construction of the required cover of $\mathrm{Mod}(\Sigma) \times \mathcal{PMF}$ is divided into two parts. For $\epsilon > 0$ the pair (g, ξ) is ϵ -thick if no geodesic along $c_{g, \xi}$ has length $< \epsilon$, and otherwise the pair is ϵ -thin.

Given a finite set $S \subset \mathrm{Mod}(\Sigma)$ we first find $\epsilon > 0$ and an S -long cover of the ϵ -thin part. Then we cover the ϵ -thick part.

When covering the thin part the main tool is the Masur-Minsky notion of subsurface projections. As in [12], the collection of all subsurfaces is divided into finitely many subcollections \mathbf{Y}^i , $i = 1, 2, \dots, k$ so that any two subsurfaces in the same subcollection overlap. There is a finite index subgroup $G < \text{Mod}(\Sigma)$ that preserves each subcollection. Our cover of the thin part will consist of k collections of sets, one for each \mathbf{Y}^i . Roughly speaking, a theorem of Rafi [60] guarantees that if (g, ξ) is in the thin part then for some subsurface Y we have a large projection distance in Y between $g(X_0)$ and ξ , and the elements of the cover will be parametrized by such subsurfaces. Here we use for each \mathbf{Y}^i the projection complex [12] to obtain a good notion of *first* subsurface with large projection for a given pair (g, ξ) , similar to the first horoball in the model case of $\text{SL}_2(\mathbb{Z})$ above. To this end we find it useful to extend the axiomatic setup for the projection complex to include projections of foliations $\xi \in \mathcal{PMF}$.

The strategy for covering the thick part is a coarse version of the model arguments recalled earlier. Here we use hyperbolicity properties of the thick part of Teichmüller space. We summarized the properties we use in the form of flow axioms. These axioms are formulated in a quasi-isometry invariant way. We note that our fellow traveler axiom (F2) is weaker than what is actually known now about Teichmüller geodesics in the thick part, see for example [21, §8].

Outline by sections. We start by listing all the axioms in Section 1. In Section 2 we construct the cover of the thin part using the projection axioms and in Section 3 we construct the cover of the thick part using the flow axioms. Section 4 contains a general discussion of the Farrell-Jones Conjecture and finite \mathcal{F} -amenability. In Section 5 we review the basics of mapping class groups: Teichmüller space, measured foliations and geodesic laminations, Teichmüller metric, quadratic differentials. In Section 6 we review the Masur-Minsky subsurface projections and verify the projection axioms. In Section 7 we verify the flow axioms. The main ingredients are Minsky's Contraction Theorem [57], the Masur Criterion [53], and Klarreich's description of the boundary of the curve complex [44]. In Section 8 we discuss Teichmüller geodesics that enter the thin part and in Section 9 we provide the roadmap showing which axioms are verified where.

The Farrell-Jones Conjecture. Surgery theory as developed by Browder-Novikov-Sullivan-Wall relates the classification of manifolds (of dimension at least 5) to algebraic invariants, i.e., to the algebraic K -groups and L -groups of $\mathbb{Z}[G]$, the integral group ring over the fundamental group of the manifold. Farrell-Jones [25] formulated a general

conjecture about the structure of these K - and L -groups. Informally, the conjecture asserts that the computation of K - and L -groups of $\mathbb{Z}[G]$ can be reduced (modulo group homology) to the computation of K - and L -groups of group rings $\mathbb{Z}[V]$, where V varies over the family VCyc of virtually cyclic subgroups of G . Often it is beneficial to consider as an intermediate step a family of subgroups \mathcal{F} containing VCyc and to consider the Farrell-Jones Conjecture relative to \mathcal{F} . The formulation of the Farrell-Jones Conjecture that we will be using is recalled in Subsection 4.B.

Building on the work of Farrell and Jones their Conjecture has been verified for a many classes of groups. Among these are hyperbolic groups, $CAT(0)$ -groups, solvable groups and lattices in Lie groups [5, 7, 40, 67, 68]. A more complete list can be found in [40, Thm. 2].

The Farrell-Jones Conjecture implies the Novikov Conjecture, but is stronger. Roughly speaking, the Novikov Conjecture asserts that a certain map is (rationally) injective whereas the Farrell-Jones Conjecture asserts it is bijective. More concretely, as reviewed below, the Farrell-Jones Conjecture implies that aspherical manifolds are determined up to homeomorphism by their fundamental group (in dimension ≥ 5), while the Novikov Conjecture only asserts that their higher signatures are determined by the fundamental group.

The surgery exact sequence. Applications to the classification of manifolds was the main motivation for the Conjecture and the monumental works surrounding it of Farrell and Jones. We give a short summary of the key instance of such an applications. In the following manifolds are always topological manifolds, i.e., we do not require a smooth or PL -structure; the topological implications of the Farrell-Jones Conjecture are cleanest. Let M be a closed oriented manifold of dimension $n \geq 5$. The simple topological structure set $\mathcal{S}(M)$ of M consists of equivalence classes $[f: X \rightarrow M]$, where X is a closed topological manifold and f is a simple homotopy equivalence. We have $[f: X \rightarrow M] = [f': X' \rightarrow M]$ if and only if there is a homeomorphism $h: X \rightarrow X'$ with $f' \circ h$ homotopic to f . Thus understanding the structure set amounts to classifying all manifolds in the simple homotopy type of M . To understand the structure set surgery theory provides the surgery exact sequence, which we outline next. Let \mathbf{L} be the L -theory spectrum of the ring \mathbb{Z} and let $\mathbf{L}\langle 1 \rangle$ be its connective cover. In particular, $\pi_n(\mathbf{L}\langle 1 \rangle) = \pi_n(\mathbf{L}) = L_n(\mathbb{Z})$ for $n \geq 1$ and $\pi_n(\mathbf{L}\langle 1 \rangle) = 0$ for $n \leq 0$. We write $L_*^s(\mathbb{Z}[G])$ for the simple L -groups of the integral group ring over the fundamental group of M . The *surgery exact sequence for topological manifolds* can be formulated

as follows

$$H_{n+1}(M; \mathbf{L}) \xrightarrow{\alpha_M} L_{n+1}^s(\mathbb{Z}[G]) \rightarrow \mathcal{S}(M) \rightarrow H_n(M; \mathbf{L}\langle 1 \rangle) \xrightarrow{\alpha'_M} L_n^s(\mathbb{Z}[G]).$$

Here $H_*(-; \mathbf{L}\langle 1 \rangle)$ and $H_*(-; \mathbf{L})$ are the homology theories associated to the corresponding spectra, α_M is Quinn's assembly map for M and α'_M is the composition of α_M with the map induced from $\mathbf{L}\langle 1 \rangle \rightarrow \mathbf{L}$. Exactness of this sequence includes the statement that the structure set $\mathcal{S}(M)$ carries a natural group structure (which is not easily described). There is an extension of this sequence to a long exact sequence. The construction and exactness of the surgery exact sequence for topological manifolds depends among other things on the work of Kirby-Siebenmann on topological manifolds and Ranicki's identification of the geometric assembly map with the algebraic assembly map. For a more detailed summary of these results see the introduction of [63].

If G is trivial, then $L_*^s(\mathbb{Z}[G]) = L_*(\mathbb{Z}) = H_*(\text{pt}; \mathbf{L})$ and α_M is the map induced by the projection $M \rightarrow \text{pt}$ for $H_*(-; \mathbf{L})$. Consequently, α_M and α'_M are both surjective. If we specialize further to $M = S^n$, then $H_n(S^n; \mathbf{L}\langle 1 \rangle) \cong L_n(\mathbb{Z})$ and $\mathcal{S}_n(S^n) = \{\text{id}_{S^n}\}$, i.e., we recover the high-dimensional Poincaré conjecture from the surgery exact sequence.

The Farrell-Jones Conjecture gives information about the L -groups in the surgery exact sequence. This information has the cleanest form if G is torsion free. So assume now that G satisfies the Farrell-Jones Conjecture and is torsion free. Then, after some algebraic manipulations, there is an isomorphism $\alpha_{BG}: H_*(BG; \mathbf{L}) \rightarrow L_n^s(\mathbb{Z}[G])$, see [49, Prop. 23]. Moreover, if $\kappa: M \rightarrow BG$ is the classifying map (inducing $\pi_1(M) = G$), then $\alpha_M = \alpha_{BG} \circ \kappa_*$. This yields an injection of the structure set $\mathcal{S}(M)$ into the relative homology group $H_{n+1}(BG, M; \mathbf{L})$ (where we think of M as a subspace of BG via κ). The cokernel of this inclusion is a subgroup of $H_n(M; L_0(\mathbb{Z})) = L_0(\mathbb{Z}) = \mathbb{Z}^1$. Moreover, as reviewed below, the K -theory part of the Farrell-Jones conjecture implies that the Whitehead group of G is trivial and therefore every homotopy equivalence $X \rightarrow M$ is simple. Consequently, the Farrell-Jones Conjecture yields an identification of all manifold structures on M with relative homology classes. Finally, let us assume in addition that M is aspherical, i.e., that κ is a homotopy equivalence. Then of course $H_{n+1}(BG, M; \mathbf{L}) = 0$ and therefore $\mathcal{S}(M) = \{[\text{id}_M]\}$. In other words, every homotopy equivalence $X \rightarrow M$ is homotopic to a homeomorphism, i.e., the Borel Conjecture holds for M .

¹In fact, under our assumptions, $H_{n+1}(BG, M; \mathbf{L})$ can be identified with the ANR -homology manifold structure set and the map to \mathbb{Z} is the Quinn obstruction [63, §25].

For the K -theory of the integral group ring $\mathbb{Z}[G]$ of a torsion free group the Farrell-Jones Conjecture predicts that the Loday assembly map $H_n(BG; \mathbf{K}_{\mathbb{Z}}) \rightarrow K_n(\mathbb{Z}[G])$ is bijective. Here $H_n(-; \mathbf{K}_{\mathbb{Z}})$ is the homology theory associated to the K -theory spectrum $\mathbf{K}_{\mathbb{Z}}$ of the integers \mathbb{Z} . Since $K_n(\mathbb{Z}) = 0$ for $n < 0$, $K_0(\mathbb{Z}) = \mathbb{Z}$, $K_1(\mathbb{Z}) = \mathbb{Z}^{\times} \cong \mathbb{Z}/2\mathbb{Z}$, the Farrell-Jones conjecture predicts for torsion free groups $K_n(\mathbb{Z}[G]) = 0$ for $n < 0$, $K_0(\mathbb{Z}[G]) = \mathbb{Z}$ and $K_1(\mathbb{Z}[G]) = \mathbb{Z}/2\mathbb{Z} \times G_{ab}$; in particular, it predicts the vanishing of the reduced projective class group $\tilde{K}_0(\mathbb{Z}[G])$ and of the Whitehead group $\text{Wh}(G)$ [49, Cor. 67]. The Farrell-Jones Conjecture has applications to a number of further conjectures about group rings, for example Kaplansky's idempotent conjecture, Bass' Conjecture about the Hattori-Stallings rank, and Serre's Conjecture that groups of type FP are of type FF [8, 48, 49].

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1. AXIOMS

Let G be a group, and let Δ be a G -space. We assume that Δ is compact, metrizable and finite dimensional. In this section we discuss somewhat elaborate axioms that will imply that the action of G on Δ is finitely \mathcal{F} -amenable. The axioms will be defined in the presence of two pieces of further data.

Definition 1.1. *Projection data* for the action of G on Δ consists of

- a finite collection of G -sets $\mathcal{Y} = \{\mathbf{Y}^1, \mathbf{Y}^2, \dots, \mathbf{Y}^k\}$;
- for each $\mathbf{Y} \in \mathcal{Y}$ and each $Y \in \mathbf{Y}$ an open subspace $\Delta(Y) \subseteq \Delta$ and a map

$$d_Y^{\pi} : (\mathbf{Y} \setminus \{Y\}) \times (\Delta(Y) \amalg \mathbf{Y} \setminus \{Y\}) \rightarrow [0, \infty].$$

We also require G -equivariance, i.e., for $g \in G$, $Y \in \mathbf{Y}$, we require $\Delta(gY) = g\Delta(Y)$ and for $X \in \mathbf{Y} \setminus \{Y\}$, $\xi \in \Delta(Y) \amalg \mathbf{Y} \setminus \{Y\}$ we require $d_{gY}^{\pi}(gX, g\xi) = d_Y^{\pi}(X, \xi)$. We also require $d_Y^{\pi}(X, Z) < \infty$ for $X, Y, Z \in \mathbf{Y}$. We will refer to the d_Y^{π} as *projection distances*.

Definition 1.2. *Flow data* for the action of G on Δ consists of

- a proper metric space T with a proper isometric G -action;

- a compact metrizable space $\overline{T} = T \sqcup \Delta$ such that $T \subseteq \overline{T}$ is open and the G -actions on T and Δ combine to a continuous G -action on \overline{T} ;
- for every compact subset $K \subseteq T$ a collection \mathcal{G}_K of (μ, A_K) -quasi-geodesic rays $c: [0, \infty) \rightarrow G \cdot K \subseteq T$.

The space \overline{T} is assumed to be finite dimensional, separable and metrizable. As indicated with the notation, the additive constant A_K for quasi-geodesic rays is uniform for all $c \in \mathcal{G}_K$, but is allowed to depend on K , while the multiplicative constant μ is required to be independent of K as well. In our application we will have $(\mu, A_K) = (1, 0)$, i.e. each $c \in \mathcal{G}_K$ will be a geodesic ray. We also require that \mathcal{G}_K is G -invariant. Finally, we require that each ray $c \in \mathcal{G}_K$ has a limit $c(t) \rightarrow c(\infty) \in \Delta$.

Remark 1.3. Examples will follow the axioms, but we point out the following. First, the projection distance $d_Y^\pi(X, Z)$ will always be finite when X, Y, Z are distinct elements of \mathbf{Y} . On the other hand, $d_Y^\pi(X, \xi)$ may be infinite or undefined when $\xi \in \Delta$. Second, the collection \mathcal{G}_K should be thought of as really depending only on $G \cdot K$ and consists of (quasi)geodesic rays in T that are contained in $G \cdot K$, which is thought of as the “thick part”.

We now fix base points $X_{\mathbf{Y}} \in \mathbf{Y}$ and a base point $x_0 \in T$. The choice of the base points only affects the values of Θ and K later on, but not the validity of the axioms.

Axiom 1.4 (Projections). For each $\mathbf{Y} \in \mathcal{Y}$ the projection distances $(d_Y^\pi)_{Y \in \mathbf{Y}}$ satisfy the following projection axioms with respect to some constant $\theta \geq 0$.

(P1) *Symmetry.* For $X, Z \in \mathbf{Y} \setminus \{Y\}$,

$$d_Y^\pi(X, Z) = d_Y^\pi(Z, X).$$

(P2) *Triangle inequality.* For all $X, Z \in \mathbf{Y} \setminus \{Y\}, \xi \in \Delta(Y) \amalg \mathbf{Y} \setminus \{Y\}$

$$d_Y^\pi(X, Z) + d_Y^\pi(Z, \xi) \geq d_Y^\pi(X, \xi).$$

(P3) *Inequality on triples.* For all $\xi \in \Delta(Y) \cap \Delta(Y') \amalg \mathbf{Y} \setminus \{Y, Y'\}$, $Y \neq Y'$ we have

$$\min\{d_Y^\pi(Y', \xi), d_{Y'}^\pi(Y, \xi)\} < \theta.$$

(P4) *Finiteness.* For all $X, Z \in \mathbf{Y}$ the set

$$\{Y \in \mathbf{Y} \mid d_Y^\pi(X, Z) > \theta\}$$

is finite.

- (P5) *Coarse semi-continuity.* For $\xi \in \Delta(Y)$, $X \in \mathbf{Y}$, $\theta < \Theta < \infty$ with $d_Y(X, \xi) \geq \Theta$ there exists a neighborhood U of ξ in Δ such that $U \subseteq \Delta(Y)$ and for all $\xi' \in U$,

$$d_Y^\pi(X, \xi') > \Theta - \theta.$$

Axiom 1.5 (Thick or thin). For any $\Theta > 0$ there is $K \subseteq T$ compact such that for any $(g, \xi) \in G \times \Delta$

- (a) either, there is $c \in \mathcal{G}_K$ with $c(0) = gx_0$ and $c(\infty) = \xi$
- (b) or, there are $\mathbf{Y} \in \mathcal{Y}$, $Y \in \mathbf{Y}$ with $\xi \in \Delta(Y)$ and $d_Y^\pi(gX_{\mathbf{Y}}, \xi) > \Theta$.

Pairs (g, ξ) to which (a) of Axiom 1.5 applies will be said to belong to the *K-thick part of $G \times \Delta$* . Pairs (g, ξ) to which (b) of Axiom 1.5 applies will be said to *admit a Θ -large projection*.

Axiom 1.6 (Flow axioms). Let $K \subseteq T$ be compact.

- (F1) *Small at ∞ .* Let $c_n \in \mathcal{G}_K$, $x_n \in T$ such that

$$d_T(\text{Im}(c), x_0) \text{ and } d_T(c_n(0), x_n)$$

are bounded. If $x_n \rightarrow \xi_- \in \Delta$, then also $c_n(0) \rightarrow \xi_-$.

- (F2) *Fellow traveling.* For any $\rho > 0$ there is $R > 0$ with the following property. For all $x \in T$, all $\xi_+ \in \Delta$, all $t \in [0, \infty)$ there exists an open neighborhood U_+ of ξ_+ in Δ with the following property. Let $c, c' \in \mathcal{G}_K$ be two quasi-geodesic rays that both start in the ρ -neighborhood of x and satisfy $c(\infty), c'(\infty) \in U_+$. We require that $d_T(c(t), c'(t)) < R$.

- (F3) *Infinite quasi-geodesics.* For all $\rho > 0$ there is $R > 0$ with the following property. For $\xi_-, \xi_+ \in \Delta$ we define $T_{K, \rho}(\xi_-, \xi_+) \subseteq T$ to consist of all x for which there are $c_n \in \mathcal{G}_K$ with $c_n(0) \rightarrow \xi_-$, $c_n(\infty) \rightarrow \xi_+$ and $d_T(\text{Im}(c_n), x) \leq \rho$. If $T_{K, \rho}(\xi_-, \xi_+) \neq \emptyset$, then we require that there exists a quasi-geodesic $c: \mathbb{R} \rightarrow T$ such that $T_{K, \rho}(\xi_-, \xi_+)$ is contained in the R -neighborhood of c . Here the additive constant for c depends only on K , while the multiplicative constant for c is required to not depend on any choices.

Remark 1.7. The fellow traveling axiom (F2) implies that for any $\alpha > 0$ there is $R > 0$ such that for all $c, c' \in \mathcal{G}_K$ with $d_T(c(0), c'(0)) \leq \alpha$ and $c(\infty) = c'(\infty)$ we have

$$\forall t \geq 0 \quad d_T(c(t), c'(t)) \leq R.$$

Remark 1.8. If T is a proper δ -hyperbolic space and Δ is its Gromov boundary, then (F1)-(F3) hold, with \mathcal{G}_K consisting of all geodesics contained in the “thick part” $G \cdot K$. Moreover, (F1)-(F2) hold when

T is a proper $CAT(0)$ space and Δ its visual boundary. However, (F3) may fail, e.g. when $T = \mathbb{R}^2$ and ξ_{\pm} are two antipodal points on the boundary circle. It is possible to weaken (F3) and only demand that $T_{K,\rho}(\xi_-, \xi_+)$ has a doubling property, see Proposition 3.5. The collection \mathcal{F} in Theorem 1.11 below would have to be suitably enlarged to include stabilizers of such sets.

Example 1.9. To fix ideas we point out that the axioms hold for the group $G = SL_2(\mathbb{Z})$ acting on the hyperbolic plane $T = \mathbb{H}^2$ with its Gromov boundary $\Delta = S^1$. Fix a pairwise disjoint and equivariant collection \mathbf{Y} of open horoballs. Projection data \mathcal{Y} will consist of the single collection \mathbf{Y} , and we will set $\Delta(Y) = \Delta$ for all $Y \in \mathbf{Y}$. If Y, Z are distinct horoballs, denote by $\pi_Y(Z)$ the nearest point projection of Z to the closure of Y . This is an open interval in the horocycle boundary of Y and it has uniformly bounded diameter. If X, Y, Z are distinct horoballs we set

$$d_Y^\pi(X, Z) = \text{diam}(\pi_Y(X) \cup \pi_Y(Z))$$

where the diameter is taken with respect to the metric in T (or if one wishes in the path metric of the horocycle; they are coarsely equivalent and the distinction is irrelevant). Similarly, if $\xi \in \Delta$ is a point on the circle that does not belong to the closure of the horoball Y , there is a well defined nearest point projection $\pi_Y(\xi)$ in the boundary of Y , and we again define

$$d_Y^\pi(X, \xi) = \text{diam}(\pi_Y(X) \cup \pi_Y(\xi))$$

Finally, if ξ is in the closure of Y , we define $d_Y^\pi(X, \xi) = \infty$ for all horoballs $X \neq Y$. Here $\pi_Y(\xi)$ is not defined, but we may think of it as the point at infinity of the horocycle.

To complete the description of the flow data, we define \mathcal{G}_K as the set of geodesic rays contained in $G \cdot K$, for any compact $K \subset T$. Verification of the axioms is left to the reader.

Example 1.10. Let G be a hyperbolic group relative to a finitely generated subgroup H (or more generally relative to a collection of such subgroups). Then G acts on a proper δ -hyperbolic space (see [18, 22, 28, 29]) with Gromov boundary Δ and the action is cocompact in the complement of a pairwise disjoint equivariant collection of horoballs. Projection and flow data can be constructed in the same way as above and all axioms hold.

The Farrell-Jones conjecture for relatively hyperbolic groups was proved in [2] and generalizing this argument to mapping class groups was the motivation for the present work.

For the description of flow and projection data in the case of mapping class groups, see Section 9.

Theorem 1.11. *Assume that there is projection data and flow data for the action of G on Δ satisfying the axioms from 1.4, 1.5 and 1.6. Assume that Δ is compact, metrizable and finite dimensional. If, in addition, \mathcal{F} contains the family VCyc of virtually cyclic subgroups of G and all isotropy groups for the action of G on all the $\mathbf{Y} \in \mathcal{Y}$, then the action of G on Δ is finitely \mathcal{F} -amenable.*

Proof. This will be an easy consequence of Theorems 2.1 and 3.1 that we prove later.

Let $S \subseteq G$ finite. Let us say that a collection of open subsets of $G \times \Delta$ is S -long at (g, ξ) if for one of its members U we have $gS \times \{\xi\} \subseteq U$.

Theorem 2.1 provides for each $\mathbf{Y} \in \mathcal{Y}$ a G -invariant collection $\mathcal{U}_{\mathbf{Y}}$ of open \mathcal{F} -subsets and a number Θ such that $\mathcal{U}_{thin} := \bigcup_{\mathbf{Y} \in \mathcal{Y}} \mathcal{U}_{\mathbf{Y}}$ is S -long at all (g, ξ) that admit a Θ -large projection. The order of each $\mathcal{U}_{\mathbf{Y}}$ is at most 1 and therefore the order of \mathcal{U}_{thin} is at most $N_{thin} := 2k - 1$.

Next we use Axiom 1.5. Thus there is $K \subseteq G$ compact such that all (g, ξ) that do not admit a Θ -large projection belong to the K -thick part of $G \times \Delta$.

Theorem 3.1 provides a G -invariant collection \mathcal{U}_{thick} of open VCyc-subsets of $G \times \Delta$ that is S -long at all (g, ξ) from the K -thick part. Moreover, the order of \mathcal{U}_{thick} is bounded by a number N_{thick} independent from K and S .

Altogether $\mathcal{U}_{thin} \cup \mathcal{U}_{thick}$ is the cover we need and the action of G on Δ is $(N_{thin} + N_{thick} + 1)$ - \mathcal{F} -amenable. \square

2. PARTIAL COVERS FROM THE PROJECTION COMPLEX

2.A. Projection covers. Throughout this section we consider a G -space Δ . We assume that we are given a G -set \mathbf{Y} , subsets $\Delta(Y) \subseteq \Delta$ for $Y \in \mathbf{Y}$ and projection distances

$$d_Y^\pi : (\mathbf{Y} \setminus \{Y\}) \times (\Delta(Y) \amalg \mathbf{Y} \setminus \{Y\}) \rightarrow [0, \infty].$$

For $\xi \in \Delta$, $X \in \mathbf{Y}$ we set $d_{\mathbf{Y}}^\pi(X, \xi) := \sup_{\xi \in \Delta(Y)} d_Y^\pi(X, \xi)$. If there is no such Y , then we set $d_{\mathbf{Y}}^\pi(X, \xi) = -\infty$. We also write $\mathcal{F}_{\mathbf{Y}}$ for the family of subgroups of G that fix an element of \mathbf{Y} .

Theorem 2.1. *Assume that the projection distances $(d_Y^\pi)_{Y \in \mathbf{Y}}$ satisfy the axioms (P1) to (P5) listed in 1.4. Pick a base point $X_{\mathbf{Y}} \in \mathbf{Y}$. Let $S \subseteq G$ be finite. Then there is $\Theta \geq 0$ and a G -invariant collection \mathcal{U} of open $\mathcal{F}_{\mathbf{Y}}$ -subsets of $G \times \Delta$ such that*

- (a) *the order of \mathcal{U} is at most 1;*

- (b) for any $(g, \xi) \in (G \times \Delta)$ with $d_Y^\pi(gX_Y, \xi) \geq \Theta$ there is $U \in \mathcal{U}$ with $gS \times \{\xi\} \subseteq U$.

The proof of Theorem 2.1 occupies the remainder of this section.

2.B. The projection complex and angles. The restrictions of the d_Y^π to $\mathbf{Y} \setminus \{Y\}$ satisfy the projection axioms (PC1) to (PC4) from [12, Sec. 3.1]. These axioms allow, for a constant $K \gg \theta$, the construction of the projection complex $\mathcal{P}_K(\mathbf{Y})$ [12, Sec. 3.3]. This complex is a graph whose set of vertices is \mathbf{Y} . We write $d_{\mathcal{P}}$ for the path metric with edges of length 1 on $\mathcal{P}_K(\mathbf{Y})$. The action of G on \mathbf{Y} extends to a simplicial action on $\mathcal{P}_K(\mathbf{Y})$. Crucial for us will be the following two properties from [12] of the projection complex.

Proposition 2.2. *There is a constant $\theta'_{\mathcal{P}} > 0$ (depending on K) such that the following holds.*

- (a) Local estimate. *If Y is an internal vertex of a geodesic in $\mathcal{P}_K(\mathbf{Y})$ from X to Z , and if X' and Z' are two vertices on this geodesic such that X' is between X and Y , and Z' is between Y and Z then $|d_Y^\pi(X, Z) - d_Y^\pi(X', Z')| < \theta'_{\mathcal{P}}$.*
- (b) Attraction property. *If $d_Y^\pi(X, Z) > \theta'_{\mathcal{P}}$, then any geodesic in $\mathcal{P}_K(\mathbf{Y})$ between X and Z will pass through Y .*

Proof. The construction of $\mathcal{P}_K(\mathbf{Y})$ depends on a bounded perturbation d_Y of the restrictions of the d_Y^π to $\mathbf{Y} \setminus \{Y\}$, i.e., $|d_Y^\pi(X, Z) - d_Y(X, Z)|$ is uniformly bounded in X, Y and Z , see [12, Thm. 3.3(B)]. Let c be a geodesic in $\mathcal{P}_K(\mathbf{Y})$ from X to Y and let X' be any internal vertex of c . Then $d_Y(X, X')$ is uniformly bounded by [12, Cor. 3.15]. Thus $d_Y^\pi(X, X')$ is also uniformly bounded and the local estimate follows from the triangle inequality.

For d_Y the attraction property is the content of the first statement of [12, Lem. 3.18]. Since d_Y is a uniformly bounded perturbation of d_Y^π , the attraction property follows for d_Y^π as well. \square

For an internal vertex Y of a geodesic c in the projection complex we define the *angle of c at Y* as $\angle_Y c := d_Y^\pi(X, Z)$ where X and Z are the two vertices on c adjacent to Y . If Y is disjoint from c , then we set $\angle_Y c = 0$; if Y is the start or end point of c , then $\angle_Y c$ remains undefined. For $X, Z \neq Y$ we set $d_Y^{\max}(X, Z) := \max\{\angle_Y c\}$, where c varies over the set of all geodesics from X to Z .

We record the following consequences of Proposition 2.2 for these the definitions. Item (d) is the main advantage d^{\max} has over d^π .

Lemma 2.3. *There is $\theta_{\mathcal{P}} > 0$ such that for all vertices $X, Z \neq Y$*

- (a) $|d_Y^\pi(X, Z) - d_Y^{\max}(X, Z)| < \theta_{\mathcal{P}}$;

- (b) for any geodesic c from X to Z we have $d_Y^{\max}(X, Z) - \mathfrak{d}_Y c < \theta_{\mathcal{P}}$;
- (c) if $d_Y^{\max}(X, Z) \geq \theta_{\mathcal{P}}$ or $d_Y^{\pi}(X, Z) \geq \theta_{\mathcal{P}}$ then any geodesic from X to Z passes through Y ;
- (d) let c be a geodesic from X to Z that passes through Y_0 and Y in this order, if

$$\max\{d_{Y_0}^{\pi}(X, Y), d_{Y_0}^{\pi}(X, Z), \mathfrak{d}_{Y_0} c\} \geq \theta_{\mathcal{P}},$$

then $d_Y^{\max}(X, Z) = d_Y^{\max}(Y_0, Z)$.

Note that there is a uniform bound on the difference between any of the three numbers appearing in the hypothesis of item (d) in the above lemma.

Proof of Lemma 2.3. For $\theta_{\mathcal{P}} > 2\theta'_{\mathcal{P}}$, properties (a) and (b) are a consequence of the local estimate provided there exists a geodesic from X to Z that passes through Y . If there is no such geodesic, properties (a) and (b) follow from the attraction property. For $\theta_{\mathcal{P}} > 3\theta'_{\mathcal{P}}$, property (c) follows again from the attraction property. Property (c) implies that under the assumption of (d) any geodesic from X to Z can be built by concatenation of a geodesic from X to Y_0 with a geodesic from Y_0 to Z . Thus (d) holds. \square

2.C. The numbers $\Theta_0, \dots, \Theta_5$.

Lemma 2.4. *For $\mathbf{S} \subset \mathbf{Y}$ finite, there is $\theta'_{\mathbf{S}} \geq 0$ such that for all $X, X' \in \mathbf{S}$, $Y, Z \in \mathbf{Y}$,*

$$|d_Y^{\pi}(X, Z) - d_Y^{\pi}(X', Z)| < \theta'_{\mathbf{S}}.$$

Proof. By finiteness, $d_Y^{\pi}(X, X')$ is bounded for $Y \in \mathbf{Y}$, $X, X' \in \mathbf{S}$. \square

Lemma 2.5. *For $\mathbf{S} \subset \mathbf{Y}$ finite, there is $\theta_{\mathbf{S}} \geq 0$ such that for $X \in \mathbf{S}$, $Y, Z \in \mathbf{Y}$, $X, Z \neq Y$ we have*

- (a) the distance between any two numbers from

$$\{d_Y^{\pi}(X', Z), d_Y^{\max}(X', Z) \mid X' \in \mathbf{S}\}$$

is $< \theta_{\mathbf{S}}$;

- (b) if $d_Y^{\max}(X, Z) \geq \theta_{\mathbf{S}}$ or $d_Y^{\pi}(X, Z) \geq \theta_{\mathbf{S}}$ then for any $X' \in \mathbf{S}$ any geodesic from X' to Z passes through Y ;
- (c) suppose there is a geodesic c from X to Z that passes through Y_0 and Y in this order, if one of the numbers

$$d_{Y_0}^{\pi}(X, Y), d_{Y_0}^{\max}(X, Y), d_{Y_0}^{\pi}(X, Z), d_{Y_0}^{\max}(X, Z)$$

is $\geq \theta_{\mathbf{S}}$, then for any $X' \in \mathbf{S}$ $d_Y^{\max}(X', Z) = d_Y^{\max}(Y_0, Z)$.

Proof. This follows by combining Lemma 2.4 with Lemma 2.3. \square

Fix now $S \subseteq G$ finite. Let $\theta_S := \theta_{S \cdot X_{\mathbf{Y}}}$ as in Lemma 2.5. Next choose numbers $0 \ll \Theta_0 \ll \Theta_1 \ll \Theta_2 \ll \Theta_3 \ll \Theta_4 \ll \Theta_5$. Later we will need estimates of the form $\Theta_i > \Theta_j + C$ for $i > j$ and for C a constant depending on θ and θ_S and it will be clear that we can choose the Θ_i at this point to satisfy all required estimates. (On the other hand, $\Theta_i := 10 \cdot (i + 1) \cdot (\theta + \theta_S)$ will certainly work.)

2.D. The finite projections $Z(g, \xi)$. For all $(g, \xi) \in G \times \Delta$ with $d_{\mathbf{Y}}^{\pi}(gX_{\mathbf{Y}}, \xi) > \Theta_4$ we pick $Z(g, \xi) \in \mathbf{Y}$ such that $d_{Z(g, \xi)}^{\pi}(gX_{\mathbf{Y}}, \xi) > \Theta_4$. In addition, if possible, choose $Z(g, \xi)$ so that $d_{Z(g, \xi)}^{\pi}(gX_{\mathbf{Y}}, \xi) > \Theta_5$. We can arrange this map to be G -equivariant, i.e., such that $Z(hg, h\xi) = hZ(g, \xi)$ for $h \in G$.

Remark 2.6. The use of the axiom of choice to produce the $Z(g, \xi) \in \mathbf{Y}$ may seem a little heavy handed. Assuming that \mathbf{Y} is countable we can do this, with a little more care, using only countable choice:

By equivariance it suffices to choose the $Z(1, \xi)$ with $1 \in G$ the unit. Fix a countable basis U_i of open sets for Δ and consider all pairs (U_i, Y) such that $d_Y^{\pi}(X_{\mathbf{Y}}, \xi) > \Theta_4$ for all $\xi \in U_i$. Choose an ordering of this countable set of pairs. Then for ξ let $Z(1, \xi) = Y$ where (U_i, Y) is the first pair with $\xi \in U_i$. By the coarse semi-continuity axiom (P5) this produces $Z(1, \xi)$ for all $(1, \xi)$ with $d_{\mathbf{Y}}^{\pi}(1, \xi) > \Theta_4 + \theta$. It is not difficult to adjust the constants in the rest of our argument to account for this slightly weaker statement.

Lemma 2.7. *Let $(g, \xi) \in G \times \Delta$ with $d_{\mathbf{Y}}^{\pi}(gX_{\mathbf{Y}}, \xi) > \Theta_5$. Then there is an open neighborhood U of ξ in Δ and $Y \in \mathbf{Y}$ such that for any $s \in S$, $\xi' \in U$ either $Z(gs, \xi') = Y$ or $d_Y^{\max}(gsX_{\mathbf{Y}}, Z(gs, \xi')) > \Theta_3$.*

Proof. It suffices to consider $g = e$. By coarse semi-continuity (P5) we find a neighborhood U of ξ in Δ such that $d_{Z(e, \xi)}^{\pi}(X_{\mathbf{Y}}, \zeta) > \Theta_5 - \theta$ for all $\zeta \in U$. For $s \in S$ then $d_{Z(e, \xi)}^{\pi}(sX_{\mathbf{Y}}, \zeta) > \Theta_5 - \theta_S - \theta > \Theta_4$ by Lemma 2.5 (a). In particular, for all $s \in S$, $\zeta \in U$, the vertex $Z(s, \zeta)$ is defined. We claim that for $(s, \zeta) \in S \times U$ with $Z(s, \zeta) \neq Z(e, \xi)$

$$(2.8) \quad \max\{d_{Z(s, \zeta)}^{\pi}(X_{\mathbf{Y}}, Z(e, \xi)), d_{Z(e, \xi)}^{\pi}(X_{\mathbf{Y}}, Z(s, \zeta))\} > \Theta_3 + \theta_S.$$

Indeed, assume $d_{Z(e, \xi)}^{\pi}(X_{\mathbf{Y}}, Z(s, \zeta)) \leq \Theta_3 + \theta_S$. Then

$$\begin{aligned} d_{Z(e, \xi)}^{\pi}(Z(s, \zeta), \zeta) &\geq d_{Z(e, \xi)}^{\pi}(X_{\mathbf{Y}}, \zeta) - d_{Z(e, \xi)}^{\pi}(X_{\mathbf{Y}}, Z(s, \zeta)) \\ &> \Theta_5 - \Theta_3 - \theta_S > \theta. \end{aligned}$$

The inequality on triples implies $d_{Z(s,\zeta)}^\pi(Z(e,\xi),\zeta) < \theta$. Thus

$$\begin{aligned} d_{Z(s,\zeta)}^\pi(X_{\mathbf{Y}}, Z(e,\xi)) &> \\ d_{Z(s,\zeta)}^\pi(sX_{\mathbf{Y}}, \zeta) - d_{Z(s,\zeta)}^\pi(X_{\mathbf{Y}}, sX_{\mathbf{Y}}) - d_{Z(s,\zeta)}^\pi(Z(e,\xi), \zeta) \\ &> \Theta_4 - \theta_S - \theta > \Theta_3 + \theta_S, \end{aligned}$$

proving (2.8). Now we combine (2.8) with Lemma 2.5 (b), (c). Thus for $(s, \zeta) \in S \times U$ with $Z(s, \zeta) \neq Z(e, \xi)$ we have

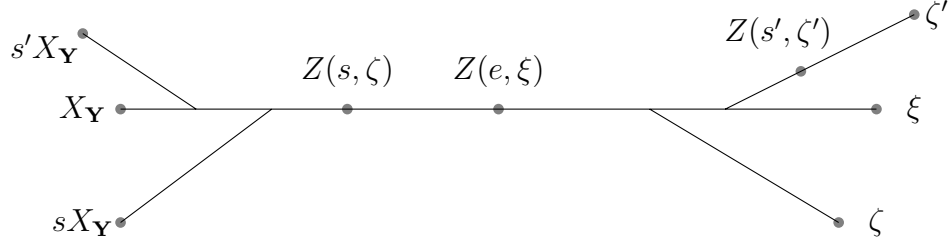
- either, for every $s' \in S$, every geodesic from $s'X_{\mathbf{Y}}$ to $Z(e, \xi)$ passes through $Z(s, \zeta)$ and $d_{Z(s,\zeta)}^{\max}(s'X_{\mathbf{Y}}, Z(e, \xi)) \geq \Theta_3$,
- or, for every $s' \in S$, every geodesic from $s'X_{\mathbf{Y}}$ to $Z(s, \zeta)$ passes through $Z(e, \xi)$ and $d_{Z(e,\xi)}^{\max}(s'X_{\mathbf{Y}}, Z(s, \zeta)) \geq \Theta_3$.

If there is no (s, ζ) to which the first case applies, then we set $Y := Z(e, \xi)$. Otherwise, we pick Y among the $Z(s, \zeta)$ to which the first case applies of minimal distance to $X_{\mathbf{Y}}$. Since this Y is also among those the $Z(s, \zeta)$ of maximal distance from $Z(e, \xi)$, it follows that it is also of minimal distance from $s'X_{\mathbf{Y}}$ for all $s' \in S$. Now, Lemma 2.4 (c) implies that whenever $Z(s, \zeta) \neq Y$, then $d_Y^{\max}(sX_{\mathbf{Y}}, Z(s, \zeta)) \geq \Theta_3$. \square

Remark 2.9. A key tool from [12] are linear orders constructed from the projection distances. For $X, Z \in \mathbf{Y}$ there is a linear order on the set of all $Y \in \mathbf{Y}$ for which $d_Y^\pi(X, Z)$ is defined and large. It is natural to add X as a minimal element and Z as a maximal element to this linear order; we will then call it the linear order from X to Z . In this order $Y < Y'$ if and only if $d_Y^\pi(X, Y')$ is large. Using these orders the construction of Y in Lemma 2.7 can be summarized as follows: For each pair (s, ζ) either $Z(s, \zeta)$ belongs to the linear order from $X_{\mathbf{Y}}$ to $Z(e, \xi)$ or $Z(e, \xi)$ belongs to the order from $X_{\mathbf{Y}}$ to $Z(s, \zeta)$. The possible positions of $Z(s, \zeta)$ in these orders are sketched in Figure 1. Then Y can be defined as the minimal element in the order from $X_{\mathbf{Y}}$ to $Z(e, \xi)$ among all $Z(s, \zeta)$ to which the first case applies. This is well defined as these linear orders are finite by the finiteness axiom (P4).

2.E. The open sets $U(Y, i)$. For $(g, \xi) \in G \times \Delta$ with $d_{\mathbf{Y}}^\pi(gX_{\mathbf{Y}}, \xi) > \Theta_4$ we now use the projection complex $\mathcal{P}_K(\mathbf{Y})$ to make the following definitions.

- For $i = 0, 1, 2$ we define vertices $Y_i(g, \xi)$ of \mathcal{P}_K as the unique vertex with the following two properties. Firstly, $d_Y^{\max}(gX_{\mathbf{Y}}, Y_i(g, \xi)) < \Theta_i$ for all $Y \in \mathbf{Y} \setminus \{gX_{\mathbf{Y}}, Y_i(g, \xi)\}$. Secondly, $Y_i(g, \xi) = Z(g, \xi)$ or there exists a geodesic c from $gX_{\mathbf{Y}}$ to $Z(g, \xi)$ with $d_{Y_i(g,\xi)}^{\max}(gX_{\mathbf{Y}}, Z(g, \xi)) \geq \Theta_i$. (The uniqueness of $Y_i(g, \xi)$ is a consequence of Lemma 2.3 since $\Theta_i > \theta_P$.)

FIGURE 1. Possible positions of $Z(s, \zeta)$

- For $Y \in \mathbf{Y}$, $i = 1, 2$ we define $U_+(Y, i) \subseteq G \times \Delta$ to consist of all (g, ξ) with $d_Y^\pi(gX_Y, \xi) > \Theta_4$ and $Y = Y_i(g, \xi)$. We define $U(Y, i)$ as the interior of $U_+(Y, i)$ in $G \times \Delta$. For $i = 1, 2$ we set $\mathcal{U}(i) := \{U(Y, i) \mid Y \in \mathbf{Y}\}$.

Lemma 2.10. *For $Y \neq Y' \in \mathbf{Y}$ we have $U(Y, i) \cap U(Y', i) = \emptyset$. For $g \in G$, $Y \in \mathbf{Y}$ we have $g(U(Y, i)) = U(gY, i)$.*

Proof. This is a direct consequence of the definition of $U(Y, i)$. \square

Lemma 2.11. *Let $(g, \xi) \in G \times \Delta$ with $d^\pi(gX_Y, \xi) > \Theta_5$. Then there are $Y \in \mathbf{Y}$ and $i \in \{1, 2\}$ with $gS \times \{\xi\} \subseteq U(Y, i)$.*

Proof. We can assume $g = e$. Set $Y_i := Y_i(e, \xi)$ for $i = 0, 1, 2$.

According to Lemma 2.7 there is $Y_3 \in \mathbf{Y}$ and an open neighborhood U of ξ in Δ such that for any $s \in S$, $\zeta \in U$, either $Y_3 = Z(s, \zeta)$ or $d_{Y_3}^{\max}(sX_Y, Z(s, \zeta)) > \Theta_3$. This implies, since $\Theta_3 > \theta_P$, by Lemma 2.3 (d),

$$(2.12) \quad d_Y^{\max}(sX_Y, Z(s, \zeta)) = d_Y^{\max}(sX_Y, Y_3)$$

for any Y on a geodesic from sX_Y to Y_3 .

We claim that for any internal vertex Y of a geodesic from Y_0 to Y_3 and any $s \in S$, $\zeta \in U$ we have

$$(2.13) \quad d_Y^{\max}(sX_Y, Z(s, \zeta)) = d_Y^{\max}(X_Y, Z(e, \xi)).$$

To prove this claim we observe first

$$d_{Y_0}^{\max}(X_Y, Y_3) > d_{Y_0}^{\max}(X_Y, Z(e, \xi)) - \theta_S \geq \Theta_0 - \theta_S > \theta_S.$$

Therefore Lemma 2.5 (c) implies $d_Y^{\max}(sX_Y, Y_3) = d_Y^{\max}(X_Y, Y_3)$ for all $s \in S$. Now (2.13) follows from (2.12).

Next we claim that, provided $Y_0 \neq Y_3$, we have

$$(2.14) \quad |d_{Y_0}^{\max}(sX_Y, Z(s, \zeta)) - d_{Y_0}^{\max}(X_Y, Z(e, \xi))| < \theta_S$$

for all $s \in S$, $\zeta \in U$. To prove this claim we note that by (2.12) to we have, $d_{Y_0}^{\max}(X_{\mathbf{Y}}, Y_3) = d_{Y_0}^{\max}(X_{\mathbf{Y}}, Z(e, \xi)) \geq \Theta_0 > \theta_S$. Thus, by Lemma 2.5 (b), any geodesic from $sX_{\mathbf{Y}}$ to Y_3 will pass through Y_0 . Using again (2.12) we have $d_{Y_0}^{\max}(sX_{\mathbf{Y}}, Z(s, \zeta)) = d_{Y_0}^{\max}(sX_{\mathbf{Y}}, Y_3)$ for $s \in S$, $\zeta \in U$. Now Lemma 2.5 (a) implies (2.14).

Let for $s \in S$, Y be an internal vertex of a geodesic from $sX_{\mathbf{Y}}$ to Y_0 . We claim that then for any $\zeta \in U$

$$(2.15) \quad d_Y^{\max}(sX_{\mathbf{Y}}, Z(s, \zeta)) < \Theta_1.$$

Suppose, by contradiction, $d_Y^{\max}(sX_{\mathbf{Y}}, Z(s, \zeta)) \geq \Theta_1$. Then, by (2.12), $d_Y^{\max}(sX_{\mathbf{Y}}, Y_3) \geq \Theta_1$. Lemma 2.5 (a) implies $d_Y^{\max}(X_{\mathbf{Y}}, Y_3) > \Theta_1 - \theta_S > \Theta_0$. Using (2.12) again, we have $d_Y^{\max}(X_{\mathbf{Y}}, Z(e, \xi)) > \Theta_0$. By definition of Y_0 and Y_3 , this implies that Y is closer to Y_3 than Y_0 . But this contradicts that Y is closer to $sX_{\mathbf{Y}}$ than Y_0 . This establishes (2.15).

Now, if $Y_0 = Y_1 = Y_2 \neq Y_3$, then, by (2.14),

$$d_{Y_0}^{\max}(sX_{\mathbf{Y}}, Z(s, \zeta)) \geq d_{Y_0}^{\max}(X_{\mathbf{Y}}, Z(e, \xi)) - \theta_S \geq \Theta_2 - \theta_S > \Theta_1.$$

Using (2.15) this implies $Y_0 = Y_1 = Y_1(s, \zeta)$ for all $s \in S$, $\zeta \in U$. Thus, in this case, $S \times U \subseteq U_+(Y_1, 1)$ and therefore $S \times \{\xi\} \subseteq U(Y_1, 1)$.

If $Y_0 \neq Y_2 \neq Y_3$ then (2.13), (2.14) and (2.15), imply $Y_2(s, \zeta) = Y_2$ for all $s \in S$, $\zeta \in U$. Thus, in this case $S \times U \subseteq U_+(Y_2, 2)$ and therefore $S \times \{\xi\} \subseteq U(Y_2, 2)$.

Finally, if $Y_2 = Y_3$, then we use in addition that $d_{Y_3}^{\max}(sX_{\mathbf{Y}}, Z(s, \zeta)) > \Theta_3$ for all $s \in S$, $\zeta \in U$ with $Z(s, \zeta) \neq Y_3$. Combining this with (2.13), (2.14) and (2.15), we find again $Y_2(s, \zeta) = Y_2$ for all $s \in S$, $\zeta \in U$. Thus, also in this case $S \times U \subseteq U_+(Y_2, 2)$ and therefore $S \times \{\xi\} \subseteq U(Y_2, 2)$. \square

Remark 2.16. Informally the key observation in the proof of Lemma 2.11 is: angles at Y_0 and Y_3 depend on (s, ζ) only up to a bounded error and all other angles behave as indicated in Figure 2.

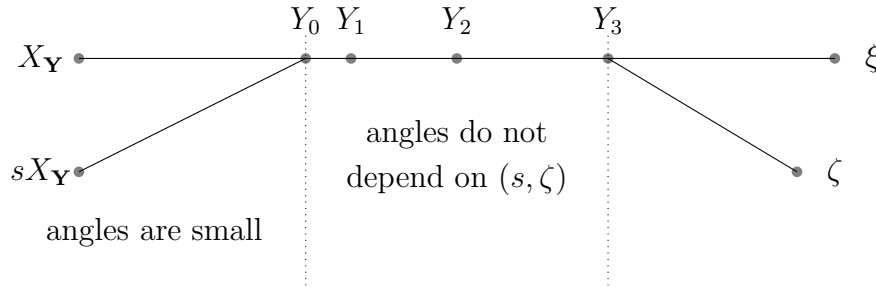


FIGURE 2. The position of the Y_i

Conclusion of Proof of Theorem 2.1. We can use $\mathcal{U} := \mathcal{U}(1) \cup \mathcal{U}(2)$ and $\Theta := \Theta_5$. Lemma 2.10 implies that the order of \mathcal{U} is at most 1 and that

its members are \mathcal{F}_Y -sets. Lemma 2.11 states that \mathcal{U} has the property required in (b) in the G -direction. \square

3. PARTIAL COVERS FROM A FLOW SPACE

Throughout this section Δ will be a finite dimensional, metrizable, separable space with a G -action. Moreover, $\overline{T} = T \cup \Delta$, \mathcal{G}_K will be flow data as in Definition 1.2. We fix a base point $x_0 \in T$ and define for $K \subseteq T$ compact

$$(G \times \Delta)_K \subseteq G \times \Delta$$

to consist of all (g, ξ) for which there exists a ray $c \in \mathcal{G}_K$ with $c(0) = gx_0$ and $c(\infty) = \xi$, i.e., $(G \times \Delta)_K$ is the K -thick part of $G \times \Delta$.

Theorem 3.1. *Assume that the flow axioms (F1), (F2) and (F3) listed in 1.6 are satisfied. Then there exists a number N_{thick} with the following property. For $S \subseteq G$ finite and $K \subseteq T$ compact, there exists a G -invariant collection \mathcal{U}_{thick} of open \mathcal{F} -subsets of $G \times \Delta$ such that the following two conditions are satisfied:*

- (a) *the order of \mathcal{U}_{thick} is at most N_{thick} ;*
- (b) *for any $(g, \xi) \in (G \times \Delta)_K$ there is $U \in \mathcal{U}_{thick}$ with $gS \times \{\xi\} \subseteq U$.*

The proof of this result is in two steps. We first build a coarse flow space that admits long thin covers, i.e., covers that have a large Lebesgue number in the direction of the flow. Then we coarsely map $(G \times \Delta)_K$ to this flow space and show that for large time the coarse flow sends $gS \times \{\xi\}$ into such long and thin sets. The cover \mathcal{U}_{thick} is then obtained by pulling back the long thin cover from the coarse flow space. To guide the reader through this proof we comment on the dependence of the appearing constants. In Theorem 3.1 we are give $K \subseteq T$ compact and $S \subseteq G$ finite. Together, K and S determine a number $\rho := d_T(K \cup Sx_0, x_0)$. Through the fellow traveler axiom (F2) ρ determines a number β in Lemma 3.10. Finally, the time τ for which the coarse flow is applied is provided in Lemma 3.11.

3.A. Coarse flow spaces. We set $V := Gx_0 \subseteq T$. Note that since the action of G on T is proper, $\overline{V} := V \cup \Delta$ is a closed and therefore compact subspace of \overline{T} . We will define the coarse flow space as a subspace of $\overline{V} \times V \times \Delta$. Informally, it consists of all triple (ξ_-, v, ξ_+) for which v coarsely belongs to a quasi geodesic from ξ_- to ξ_+ .

Definition 3.2. For $K \subseteq T$ compact and $\rho > 0$ we define $CF_0(K, \rho)$ be the subspace of $V \times V \times \Delta$ consisting of all triples $(v_-, v, \xi_+) \in V \times V \times \Delta$ for which there is $c \in \mathcal{G}_K$ with $d_T(c(0), v_-) \leq \rho$, $d_T(\text{Im}(c), v) \leq \rho$ and $c(\infty) = \xi_+$.

We define the *coarse* (K, ρ) -flow space $CF(K, \rho)$ as the closure of $CF_0(K, \rho)$ inside of $\overline{V} \times V \times \Delta$. For $(x_-, \xi_+) \in \overline{V} \times \Delta$ we define the *coarse flow line* between x_- and ξ_+ as

$$V_{K,\rho}(x_-, \xi_+) := \{v \in V \mid (x_-, v, \xi_+) \in CF(K, \rho)\} \subseteq V.$$

Lemma 3.3. *For $K \subseteq T$ compact and $\rho > 0$ there is $R > 0$ with the following property.*

- (a) *Let $(v_-, \xi_+) \in V \times \Delta$ with $V_{K,\rho}(v_-, \xi_+) \neq \emptyset$. Then there exists a quasi-geodesic ray $c: [0, \infty) \rightarrow T$ such that the coarse flow line $V_{K,\rho}(v_-, \xi_+)$ is contained in the R -neighborhood of the image of c .*
- (b) *Let $(\xi_-, \xi_+) \in \Delta \times \Delta$ with $V_{K,\rho}(\xi_-, \xi_+) \neq \emptyset$. Then there exists a quasi-geodesic $c: \mathbb{R} \rightarrow T$ such that the coarse flow line $V_{K,\rho}(\xi_-, \xi_+)$ is contained in the R -neighborhood of the image of c .*

Here the additive constant for the quasi-geodesic (ray) c depends only on K and ρ , while the multiplicative constant is independent from K and ρ .

Proof. (a) We use R from the fellow traveling axiom (F2). Still from (F2) we obtain for all $t \in [0, \infty)$ a neighborhood W_t of ξ_+ in Δ such that for all $c, c' \in \mathcal{G}_K$ with $d_T(c(0), v_-), d_T(c'(0), v_-) \leq \rho$ and $c(\infty), c'(\infty) \in W_t$ we have $d_T(c(t), c'(t)) \leq R$. After increasing constants, if necessary, we can assume W_t is constant on intervals $[n, n+1)$, $n \in \mathbb{N}$. Thus we can also assume, that for $t \geq t'$ we have $W_t \subseteq W_{t'}$. If $V_{K,\rho}(v_-, \xi_+) \neq \emptyset$, then for $t \geq 0$ there is $c_t \in \mathcal{G}_K$ with $d_T(c_t(0), v_-) \leq \rho$ and $c_t(\infty) \in W_t$. We now define the quasi-geodesic ray c by $c(t) := c_t(t)$. The multiplicative constant for c agrees with the multiplicative constant of the rays from \mathcal{G}_K , while the additive constant may increase by at most R . It is not difficult to check that $V_{K,\rho}(v_-, \xi_+)$ is contained in the $R + \rho$ -neighborhood of the image of c .

(b) The quasi-geodesic c and R are provided by the infinite quasi-geodesic axiom (F3). Using in addition the small at ∞ axiom (F1) it follows that $V_{K,\rho}(\xi_-, \xi_+)$ is contained in the R -neighborhood of the image of c . \square

Lemma 3.4. *Let $K \subseteq T$ compact and $\rho > 0$. Then for any $(x_-, \xi_+) \in \overline{V} \times \Delta$ there exists a quasi isometric embedding $V_{K,\rho}(x_-, \xi_+) \rightarrow \mathbb{Z}$. Moreover, the additive constant for this embedding depends only on K and ρ , while the multiplicative constant is also independent from K .*

Proof. This follows from Lemma 3.3 since the R -neighborhood of a quasi-geodesic ray is quasi-isometric to $\mathbb{N} \subset \mathbb{Z}$ and the R -neighborhood of a quasi-geodesic is quasi-isometric to \mathbb{Z} . \square

3.B. Long thin covers. A subset $W \subseteq V$ is said to be R -separated if $d_T(w, w') \geq R$ for all $w \neq w' \in W$. A subset $V_0 \subset V$ is said to be (D, R_0) -doubling if the following holds for all $R \geq R_0$: if $W \subseteq V_0$ is R -separated and contained in a ball of radius $2R$, then the cardinality of W is at most D .

Proposition 3.5. *Let $K \subseteq T$ be compact and $\rho > 0$.*

- (a) $\dim CF(K, \rho) \leq 2 \dim \Delta < \infty$.
- (b) *For all $(x_-, \xi_+) \in \overline{V} \times \Delta$ the set $V_{K, \rho}(x_-, \xi_+)$ is (D, R_0) -doubling. Here the constant D is independent of K , ρ , x_- and ξ_+ , while the constant R_0 depends on K and ρ , but not on x_- and ξ_- .*
- (c) *For each $(x_-, v, \xi_+) \in CF(K, \rho)$, the isotropy group $G_{(x_-, \xi_+)} := \{g \in G \mid g(x_-, \xi_+) = (x_-, \xi_+)\}$ of $(x_-, \xi_+) \in \overline{V} \times \Delta$ is virtually cyclic.*

Proof. (a) As \overline{V} and Δ are separable and metrizable any subspace of $\overline{V} \times V \times \Delta$ is of dimension at most $\dim \overline{V} + \dim \Delta = 2 \dim \Delta$.

(b) The metric space \mathbb{Z} is (D, R_0) -doubling with $D = 3$, $R_0 = 0$. It follows that every space that quasi isometrically embeds into \mathbb{Z} is (D', R'_0) -doubling with D' depending on D and the multiplicative constant of the quasi isometry and R'_0 depending on D, R_0 and the constants for the quasi isometry. Therefore (b) is a consequence of Lemma 3.4.

(c) The isotropy group G_{x_-, ξ_+} acts properly and isometrically on the coarse flow line $V_{L, \rho}(x_-, \xi_+)$. Since $V_{L, \rho}(x, \xi)$ embeds quasi-isometrically into \mathbb{Z} by Lemma 3.4 it follows that $G_{x, \xi}$ is virtually cyclic. \square

We can now prove that our coarse flow spaces admit long thin covers.

Proposition 3.6. *There is a number N_{long} such that for any $K \subseteq T$ compact and any $\rho, \beta > 0$ there exists a G -invariant cover $\mathcal{U}_{\text{long}}$ of $CF(K, \rho)$ by open VCyc-sets such that the following two conditions are satisfied:*

- (a) *the order of $\mathcal{U}_{\text{long}}$ is at most N_{long} ;*
- (b) *for any $(x_-, v, \xi_+) \in CF(K, \rho)$ there is $U \in \mathcal{U}_{\text{long}}$ with*

$$\{x_-\} \times B_\beta(v) \times \{\xi_+\} \cap CF(K, \rho) \subseteq U.$$

Here $B_\beta(v)$ is the β -neighborhood of v in V .

Proof. Using Proposition 3.5 this follows from [2, Thm. 1.1]. \square

If $U \in \mathcal{U}_{long}$ as above, then, typically, as a subset of $\overline{V} \times V \times \Delta$ the set U will be very small (i.e. thin) in the \overline{V} - and the Δ -coordinate, while the coordinate in V varies over a subset that is long in the coarse flow lines (and thus long and coarsely thin).

3.C. The coarse flow. Informally, we have a coarse flow on the coarse flow space that moves towards ξ_+ along coarse flow lines. The partially defined coarse maps ι_τ in the next definition should be thought of as the composition of the coarse flow for time τ with the map ι_0 that sends (g, ξ) to the initial point gx_0 in the coarse flow line $V_{K,\rho}(gx_0, \xi)$.

Definition 3.7. Let $K \subseteq T$ compact and $\rho \geq 0$. For $\tau \geq 0$ and $(g, \xi) \in G \times \Delta$ we define

$$\iota_\tau(g, \xi) \subseteq V_{K,\rho}(gx_0, \xi)$$

to consist of all $v \in V$ for which there is $c \in \mathcal{G}_K$ with $d_T(c(0), gx_0) \leq \rho$, $d_T(c(\tau), v) \leq \rho$ and $c(\infty) = \xi$.

For $K \subseteq T$ compact and $S \subseteq G$ finite we enlarge $(G \times \Delta)_K$ to

$$(G \times \Delta)_K^S := \{(gs, \xi) \mid s \in S, (g, \xi) \in (G \times \Delta)_K\}$$

in order to have a space that contains $gS \times \{\xi\}$ for all $(g, \xi) \in (G \times \Delta)_K$. We now use ι_τ to pull back open sets from the coarse flow space $CF(K, \rho)$ to $G \times \Delta$ and to open subsets of $(G \times \Delta)_K^S$.

Definition 3.8. Let $K \subseteq T$ compact and $\rho \geq \text{diam } K \cup \{x_0\}$. For $U \subseteq CF(K, \rho)$ and $\tau > 0$ we define $\iota^{-\tau}U \subseteq G \times \Delta$ to consist of all (g, ξ) for which

$$\{gx_0\} \times \iota_\tau(g, \xi) \times \{\xi\} \subseteq U$$

For $S \subseteq G$ finite we define

$$\iota_S^{-\tau}U \subseteq (G \times \Delta)_K^S$$

as the interior of $\iota^{-\tau}U \cap (G \times \Delta)_K^S$ in $(G \times \Delta)_K^S$. If \mathcal{U} is a collection of open subsets of $CF(K, \rho)$, then we set $\iota_S^{-\tau}\mathcal{U} := \{\iota_S^{-\tau}U \mid U \in \mathcal{U}\}$.

In the proof of Theorem 3.1 we will later use a collection of the form $\iota_S^{-\tau}\mathcal{U}_{long}$, where \mathcal{U}_{long} comes from Proposition 3.6.

The intersection with $(G \times \Delta)_K^S$ in the definition of $\iota_S^{-\tau}U$ is used to guarantee $\iota_\tau(g, \xi) \neq \emptyset$ in the proof of the following lemma.

Lemma 3.9. Let $K \subseteq T$ compact, $S \subseteq G$ finite. Let $\rho \geq d_T(K \cup Sx_0, x_0)$. Let $U, U' \subseteq CF(K, \rho)$, $\tau > 0$ and $\gamma \in G$. Then

- (a) if $U \cap U' = \emptyset$, then $\iota_S^{-\tau}U \cap \iota_S^{-\tau}U' = \emptyset$;
- (b) $\iota_S^{-\tau}(\gamma U) = \gamma(\iota_S^{-\tau}U)$.

Proof. We start with (a). Let $(g, \xi) \in \iota_S^{-\tau} U \cap \iota_S^{-\tau} U'$. Then $(g, \xi) \in (G \times \Delta)_K^S$ and $\{gx_0\} \times \iota_\tau(g, \xi) \times \{\xi\} \subseteq U \cap U'$. It remains to show that $\iota_\tau(g, \xi) \neq \emptyset$. Since $(g, \xi) \in (G \times \Delta)_K^S$ there are $s \in S$ and $c \in \mathcal{G}_K$ with $c(0) = gsx_0$, $c(\infty) = \xi$. There is $h \in G$ such that $c(\tau) \in hK$, since $c \in \mathcal{G}_K$. Now $\rho \geq d_T(K, x_0)$ implies $d_T(c(\tau), hx_0) \leq \rho$, and $\rho \geq d_T(Sx_0, x_0)$ implies $d_T(c(0), gx_0) = d_T(gsx_0, gx_0) \leq \rho$. Thus $hx_0 \in \iota_\tau(g, \xi)$.

Assertion (b) is a direct consequences of the definitions. \square

3.D. Construction of \mathcal{U}_{thick} . By construction, the coarse flow lines are thickenings of the quasi geodesics from \mathcal{G}_K . In the next lemma we replace one of the two quasi geodesics appearing in the fellow traveler axiom (F2) with a coarse flow line.

Lemma 3.10. *Let $K \subseteq T$ be compact and $S \subseteq G$ be finite. Let $\rho \geq d_T(Sx_0, x_0)$. There is $\beta > 0$ such that the following holds.*

Let $c \in \mathcal{G}_K$ and $\tau \geq 0$ and such that $c(\tau) \in K$. Then there exists a neighborhood W of $\xi := c(\infty)$ such that for all $\xi' \in W$, $g' \in G$, $v' \in \iota_\tau(g', \xi')$ with $d_T(c(0), g'x_0) \leq \rho$ we have

$$d_T(x_0, v') \leq \beta.$$

Proof. We apply the fellow traveler axiom (F2) for 2ρ and obtain a number $R \geq 0$. Still from (F2) we obtain for $x := c(0)$, $\xi := c(\infty)$, and $t := \tau$ a neighborhood W of ξ . Let now $\xi' \in W$, $g' \in G$, $v' \in \iota_\tau(g'x_0, \xi')$ with $d_T(c(0), g'x_0) \leq \rho$. Then there is $c' \in \mathcal{G}_K$ with $d_T(c'(0), g'x_0) \leq \rho$, $d_T(c'(\tau), v') \leq \rho$ and $c'(\infty) = \xi'$. Now $d_T(c(0), c'(0)) \leq d_T(c(0), g'x_0) + d_T(g'x_0, c'(0)) \leq 2\rho$. Therefore, the assertion of the fellow traveling property yields $d_T(c(\tau), c'(\tau)) \leq R$. This implies

$$\begin{aligned} d_T(x_0, v') &\leq d_T(x_0, c(\tau)) + d_T(c(\tau), c'(\tau)) + d_T(c'(\tau), v') \\ &\leq \text{diam } K \cup \{x_0\} + R + \rho =: \beta. \end{aligned}$$

\square

To prove Theorem 3.1 we will use a β -long thin cover \mathcal{U}_{long} from Proposition 3.6 where β comes from Lemma 3.10. In the next lemma we use the small at ∞ axiom (F1) to show that for sufficiently large τ the pull back of \mathcal{U}_{long} with ι_τ to $(G \times \Delta)_K^S$ is S -long for all $(g, \xi) \in (G \times \Delta)_K$, i.e., it satisfies the assertion (b) in Theorem 3.1.

Lemma 3.11. *Let $K \subseteq T$ be compact and $S \subseteq G$ be finite. Let $\rho \geq d_T(Sx_0, x_0)$. Let β be as in Lemma 3.10. Let \mathcal{U}_{long} be the cover of $CF(K, \rho)$ appearing in Proposition 3.6. Then there is $\tau > 0$ such that for all $(g, \xi) \in (G \times \Delta)_K$ there is $U \in \mathcal{U}_{long}$ with*

$$gS \times \{\xi\} \subseteq \iota_S^{-\tau} U.$$

Proof. We argue by contradiction and assume that the assertion fails. Then, for $\tau \rightarrow \infty$, there are $(g_\tau, \xi_\tau) \in (G \times \Delta)_K$ such that

$$(3.12) \quad g_\tau S \times \{\xi_\tau\} \not\subseteq \iota_S^{-\tau} U \quad \text{for all } U \in \mathcal{U}_{long}.$$

Since $(g_\tau, \xi_\tau) \in (G \times \Delta)_K$ there is $c_\tau \in \mathcal{G}_K$ with $c_\tau(0) = g_\tau x_0$ and $c_\tau(\infty) = \xi_\tau$. Since \mathcal{U}_{long} is G -invariant, we may assume that $c_\tau(\tau) \in K$ for all τ . In particular, $x_0 \in V_{K,\rho}(g_\tau x_0, \xi_\tau)$ for all τ .

Since \overline{V} and Δ are compact we can, after a subsequence, assume that

$$\lim_{\tau \rightarrow \infty} (g_\tau x_0, x_0, \xi_\tau) = (\xi_-, x_0, \xi_+) \in CF(K, \rho)$$

exists. By Proposition 3.6 there is $U \in \mathcal{U}_{long}$ such that

$$\{\xi_-\} \times B_\beta(x_0) \times \{\xi_+\} \cap CF(K, \rho) \subseteq U.$$

As U is open and $B_\beta(x_0)$ is finite there are open neighborhoods $U_- \subseteq \overline{V}$ of ξ_- and $U_+ \subseteq \Delta$ of ξ_+ such that

$$(3.13) \quad U_- \times B_\beta(x_0) \times U_+ \cap CF(K, \rho) \subseteq U.$$

The small at ∞ axiom (F1) implies that for all $s \in S$ eventually $g_\tau s x_0 \in U_-$. We find now τ such that $g_\tau s x_0 \in U_-$ for all $s \in S$ and $\xi_\tau \in U_+$. We claim that

$$(3.14) \quad g_\tau S \times \{\xi_\tau\} \subseteq \iota_S^{-\tau} U.$$

This will contradict (3.12). To prove (3.14) we apply Lemma 3.10 to $c := c_\tau$ and obtain a neighborhood W of ξ_τ in Δ . After shrinking W we may assume $W \subseteq U_+$. We claim that

$$(3.15) \quad g_\tau S \times W \cap (G \times \Delta)_K^S \subseteq \iota^{-\tau} U$$

which will imply (3.14). Let $s \in S$ and $\xi' \in W$ with $(g_\tau s, \xi') \in (G \times \Delta)_K^S$. We need to show that

$$\{g_\tau s x_0\} \times \iota_\tau(g_\tau s, \xi') \times \{\xi'\} \subseteq U.$$

Let $v' \in \iota_\tau(g_\tau s, \xi')$. Since $\xi' \in W$ and since

$$d_T(c_\tau(0), g_\tau s x_0) = d_T(g_\tau x_0, g_\tau s x_0) \leq \rho$$

we can use the assertion of Lemma 3.10 (for $g' := g_\tau s$) to obtain $d_T(x_0, v') \leq \beta$. Since $g_\tau s x_0 \in U_-$ and since $\xi' \in W \subseteq U_+$ we obtain $(g_\tau s x_0, v', \xi') \in U$ from (3.13). Thus (3.15) holds. \square

Proof of Theorem 3.1. Set $N_{thick} := N_{long}$, where N_{long} is from Proposition 3.6. Let $S \subseteq G$ finite and $K \subseteq T$ compact be given. Let $\rho := d_T(K \cup S x_0, x_0)$. Let β as in Lemma 3.10. Let \mathcal{U}_{long} be the cover of $CF(K, \rho)$ appearing in Proposition 3.6. As \mathcal{U} consists of VCyc-subsets and is G -invariant, the same holds for $\iota_S^{-\tau} \mathcal{U}_{long}$ for any τ by Lemma 3.9. Lemma 3.9 also implies that for any τ , the order of $\iota_S^{-\tau} \mathcal{U}_{long}$, does not

exceed the order of \mathcal{U}_{long} . By definition $\iota_S^{-\tau}$ consists of open subsets of $(G \times \Delta)_K^S$. As Δ is metrizable there exists a G -invariant metric on $G \times \Delta$. This allows us to extend each $V \in \iota_S^{-\tau} \mathcal{U}$ to an open subset $V' \subseteq G \times \Delta$ such that $\mathcal{U}_{thick} := \{V' \mid V \in \iota_S^{-\tau} \mathcal{U}\}$ also consists of VCyc-subsets, also is G -invariant and also is of order at most N_{thick} , see Lemma 3.16 below for more details. Finally, there is, by Lemma 3.11, $\tau > 0$ such that for each $(g, \xi) \in (G \times \Delta)_K$ there is $V = \iota_S^{-\tau} U$ with

$$gS \times \{\xi\} \subseteq V \subseteq V' \in \mathcal{U}_{thick}.$$

□

Lemma 3.16. *Let X be a G -space and Y be a G -invariant subspace. Assume that the topology on X can be generated by a G -invariant metric d . Let \mathcal{U} be a G -invariant collection of open \mathcal{F} -subsets of Y . Then there exists a G -invariant collection \mathcal{U}^+ of open \mathcal{F} -subsets of X such that*

- (a) *the order of \mathcal{U}^+ equals the order of \mathcal{U} ;*
- (b) *for each $U \in \mathcal{U}$ there is $U^+ \in \mathcal{U}^+$ with $U = U^+ \cap Y$.*

Proof. For $U \in \mathcal{U}$ set $U^+ := \{x \in X \mid d(x, U) < d(x, Y \setminus U)\}$. It is not difficult to check that $\mathcal{U}^+ := \{U^+ \mid U \in \mathcal{U}\}$ has the required properties, see for example [2, App. B]. □

4. FINITELY \mathcal{F} -AMENABLE ACTIONS.

4.A. Finite extensions and N - \mathcal{F} -amenability. The main results in this section are Propositions 4.4 and 4.5. While in principle it is possible to prove these directly from the definition, we find it convenient to reformulate N - \mathcal{F} -amenability in terms of maps to simplicial complexes and prove the statements from this point of view. The covers one would naturally write down from the definition would not be \mathcal{F} -covers, corresponding to actions on simplicial complexes with elements that leave a simplex invariant without fixing it pointwise. This is fixed by barycentrically subdividing, while the operation on covers is less transparent.

Let E be a simplicial complex with vertex set $V(E)$. Every point of E can be written as $y = \sum_{v \in V(E)} y_v \cdot v$ with $y_v \in [0, 1]$ and $\sum_{v \in V(E)} y_v = 1$. The ℓ^1 -metric on E is defined by $d_E^1(y, y') := \sum_{v \in V(E)} |y_v - y'_v|$. We briefly discuss products of simplicial complexes. Let E be a simplicial complex. For $n \in \mathbb{N}$ we define a simplicial structure on the N -fold cartesian product $E^{\times n}$ of E as follows: First we replace E by its barycentric subdivision. This is a locally ordered simplicial complex; for each simplex the set of its vertices has a linear order and this order

is compatible with taking faces of simplices. The product of locally ordered simplicial complexes is canonically a simplicial complex. For a simplicial action of a group G on E the product action on $E^{\times n}$ is also simplicial. On $E^{\times n}$ it will be convenient to use the product metric $d_{E^{\times n}}$ defined by $d_{E^{\times n}}((y_1, \dots, y_n), (y'_1, \dots, y'_n)) = \max_{1 \leq i \leq n} d_E^1(y_i, y'_i)$. This is not the ℓ^1 -metric $d_{E^{\times n}}^1$, but the change is uniformly controlled, provided that E is finite dimensional.

Lemma 4.1. *Fix $n, N \in \mathbb{N}$. Then for any $\varepsilon > 0$ there is $\varepsilon_0 > 0$ such that for any simplicial complex E of dimension at most N we have for all $y = (y_1, \dots, y_n), y' = (y'_1, \dots, y'_n) \in E^{\times n}$*

$$\begin{aligned} d_{E^{\times n}}(y, y') < \varepsilon_0 &\implies d_{E^{\times n}}^1(y, y') < \varepsilon, \\ d_{E^{\times n}}^1(y, y') < \varepsilon_0 &\implies d_{E^{\times n}}(y, y') < \varepsilon. \end{aligned}$$

Proof. If $E = \Delta^{N'}$, then this is a consequence of compactness of $(\Delta^{N'})^{\times n}$. But this case implies the general case for the following reason. Let $y, y' \in E^{\times n}$ be given. Then there are simplices σ_i, σ'_i , $i = 1, \dots, n$ of E with $y_i \in \sigma_i$, $y'_i \in \sigma'_i$. Let $F \subset E$ be the subcomplex spanned by the σ_i . This subcomplex has at most $2n(N+1)$ -many vertices and embeds therefore into an N' -simplex $\sigma \cong \Delta^{N'}$, where $N' = 2n(N+1) - 1$. Since $d_{E^{\times n}}(y, y') = d_{F^{\times n}}(y, y') = d_{\sigma^{\times n}}(y, y')$ and $d_{E^{\times n}}^1(y, y') = d_{F^{\times n}}^1(y, y') = d_{\sigma^{\times n}}^1(y, y')$, the general case follows. \square

Let E be a simplicial complex equipped with a simplicial G -action and Δ be a G -space. For $S \subset G$ finite and $\varepsilon > 0$ a continuous map $f: \Delta \rightarrow E$ is said to be (S, ε) -equivariant if

$$\sup_{x \in \Delta, s \in S} d_E^1(sf(x), f(sx)) < \varepsilon.$$

Let \mathcal{F} be a family of subgroups of G . By an (G, \mathcal{F}) -simplicial complex we mean a simplicial complex E with a simplicial G -action such that all isotropy groups belong to \mathcal{F} .

Lemma 4.2. *Let \mathcal{F} be a family of subgroups of G and Δ be a compact metrizable space with a G -action. Then the following are equivalent.*

- (a) *The action of G on Δ is N - \mathcal{F} -amenable;*
- (b) *For any $S \subseteq G$ finite and $\varepsilon > 0$ there exists a (G, \mathcal{F}) -simplicial complex of dimension at most N and an (S, ε) -equivariant map $\Delta \rightarrow E$.*

Proof. This is proven in [30, Prop. 4.2] with the following two minor changes: Firstly, in [30] the covers of $G \times \Delta$ are in addition assumed to be cofinite for the action of G . However, since Δ is compact, it is always possible to pass to a cofinite subcover. Secondly, in [30]

the family \mathcal{F} is assumed to be closed under taking supergroups of finite index. This has the advantage that the isotropy groups for a simplicial action belong to \mathcal{F} if and only if the isotropy groups of all vertices belong to \mathcal{F} . For general \mathcal{F} this is only true for cellular actions. However, the induced action on the nerve of an \mathcal{F} -cover is cellular. Therefore [30, Prop. 4.2] remains true without the assumption that \mathcal{F} is closed under taking supergroups of finite index. \square

We will use Lemma 4.2 to discuss the behavior of $N\text{-}\mathcal{F}$ -amenability under finite extensions. This is closely related to [9, Sec. 5]. As a preparation we discuss coinduction. Let $G_0 \subset G$ be a subgroup of finite index. For a G_0 -space E_0 we set $E := \text{map}_{G_0}(G, E_0)$; this is the coinduction of E_0 from G_0 to G . We obtain an action of G on E as follows: the action of $g \in G$ on $(y: G \rightarrow E_0) \in E$ is given by the formula $(gy)(a) := y(ag)$ for all $a \in G$. If G_y is the isotropy subgroup of G for $(y: G \rightarrow E_0) \in E$, then $G_y \cap G_0$ is contained in the isotropy subgroup $(G_0)_{y(e)}$ of G_0 for $y(e) \in E_0$. If E_0 is a simplicial complex, then E is a simplicial complex with the following construction. First we replace E_0 by its barycentric subdivision. Now it is a locally ordered simplicial complex and the action of G_0 preserves this order. Write $V(E_0)$ for the vertices of E_0 and define the vertices of E by $V(E) := \text{map}_{G_0}(G, V(E_0))$. Now $v_0, \dots, v_n \in V(E)$ span a simplex for E if for each $g \in G$ the vertices $v_0(g), \dots, v_n(g)$ span a simplex (possibly of dimension $< n$) of E_0 and in the local order of this simplex we have $v_0(g) \leq v_1(g) \leq \dots \leq v_n(g)$. It will be convenient to use the G -invariant metric d_E on E defined by $d_E(y, y') := \max_{a \in G} d_{E_0}^1(y(a), y'(a))$. This is *not* the ℓ^1 -metric on E . However, since, forgetting the G -action, $E = E_0^{\times m}$ for $m = [G : G_0]$, Lemma 4.1 implies that if E_0 is finite dimensional, then the identity on E is in both directions between (E, d_E^1) and (E, d_E) uniformly continuous.

If $f_0: \Delta_0 \rightarrow E_0$ is G_0 -equivariant, then we obtain a G -map

$$\text{map}_{G_0}(G, \Delta_0) \rightarrow \text{map}_{G_0}(G, E_0), \quad \xi \mapsto f_0 \circ \xi.$$

But if f_0 is not G_0 -equivariant (maybe only (S, ε) -equivariant), then this only defines a map $\text{map}_{G_0}(G, \Delta_0) \rightarrow \text{map}(G, E_0)$. If $G = G_0 t_1 \sqcup \dots \sqcup G_n t_n$, then there is a projection $\pi: \text{map}(G, E_0) \rightarrow \text{map}_{G_0}(G, E_0)$ determined by $\pi(y)(t_i) = y(t_i)$ for $i = 1, \dots, n$. We write now

$$f: \text{map}_{G_0}(G, \Delta_0) \rightarrow \text{map}_{G_0}(G, E_0)$$

for the map $\pi \circ (f_0)_*$; it is determined by $(f(\xi))(t_i) = f_0(\xi(t_i))$ for $i = 1, \dots, n$.

Lemma 4.3. *For any finite $S \subset G$ and $\varepsilon > 0$ there are $S_0 \subset G_0$ finite and $\varepsilon_0 > 0$ such that the following holds. Suppose that $f_0: \Delta_0 \rightarrow E_0$ is (S_0, ε_0) -equivariant where $\dim E_0 \leq N$, then $f: \text{map}_{G_0}(G, \Delta_0) \rightarrow \text{map}_{G_0}(G, E_0) =: E$ with $(f(\xi))(t_i) = f_0(\xi(t_i))$ for $i = 1, \dots, n$ (S, ε) -equivariant.*

Proof. Because of Lemma 4.1 we can use the metric d_E instead of d_E^1 . Note that $d_E(y, y') = \max_i d_{E_0}^1(y(t_i), y'(t_i))$, since the action of G_0 on E_0 is isometric for $d_{E_0}^1$.

Let $S \subset G$ be finite. Set $S_0 := \{t_i s t_j^{-1} \mid s \in S, 1 \leq i, j \leq n\} \cap G_0$. Let $f_0: \Delta_0 \rightarrow E_0$ be (S_0, ε) -equivariant. For $s \in S$, $\xi \in \text{map}_{G_0}(G, \Delta_0)$ and t_i we pick t_j with $t_i s \in G_0 t_j$, thus $t_i s t_j^{-1} \in G_0$. Then

$$\begin{aligned} (f(s\xi))(t_i) &= f_0((s\xi)(t_i)) = f_0(\xi(t_i s)) \\ &= f_0(\xi(t_i s t_j^{-1} t_j)) = f_0(t_i s t_j^{-1} \xi(t_j)) \\ (s(f(\xi)))(t_i) &= (f(\xi))(t_i s) = (f(\xi))(t_i s t_j^{-1} t_j) \\ &= t_i s t_j^{-1} (f(\xi)(t_j)) = t_i s t_j^{-1} f_0(\xi(t_j)) \end{aligned}$$

and therefore $d_{E_0}^1((f(s\xi))(t_i), (s(f(\xi)))(t_i)) < \varepsilon$. It follows that

$$d_E(f(s\xi), s(f(\xi))) < \varepsilon$$

and that f is (S, ε) -equivariant. \square

Proposition 4.4. *Let G act on the compact metrizable space Δ . Let G_0 be a subgroup of finite index n in G and let \mathcal{F}_0 be a family of subgroups of G_0 . Suppose that the restriction of the action on Δ to the subgroup G_0 is N - \mathcal{F}_0 -amenable. Then the action of G on Δ is $n \cdot N$ - \mathcal{F} -amenable, where \mathcal{F} is the family of subgroups F of G for which $F \cap G_0$ belongs to \mathcal{F}_0 .*

Proof. It suffices to show that condition (b) from Lemma 4.2 passes from G_0 to G . Let $S \subseteq G$ be finite and $\varepsilon > 0$ be given. Write Δ_0 for Δ with the action restricted to G_0 . Let $f_0: \Delta_0 \rightarrow E_0$ be (S_0, ε_0) -equivariant where E_0 is an (G_0, \mathcal{F}_0) -simplicial complex of dimension at most N and $S_0 \subseteq G_0$ finite and $\varepsilon_0 > 0$ come from Lemma 4.3. Let $E := \text{map}_{G_0}(G, E_0)$; this is an (G, \mathcal{F}) -simplicial complex of dimension $\leq n \cdot N$. By Lemma 4.3 there exists an (S, ε) -equivariant map $f: \text{map}_{G_0}(G, \Delta_0) \rightarrow E$. Composing f with the G -equivariant map $\Delta \rightarrow \text{map}_{G_0}(G, \Delta_0)$, $\xi \mapsto (g \mapsto g\xi)$ we obtain a (S, ε) -equivariant map $\Delta \rightarrow E$. \square

Proposition 4.5. *Let G_0 be a subgroup of G of finite index n . Let \mathcal{F}_0 be a family of subgroups of G_0 . Let \mathcal{F} be the family of subgroups F of G for which $F \cap G_0$ belongs to \mathcal{F}_0 . Assume that there exists an*

$N\text{-}\mathcal{F}_0$ -amenable action of G_0 on a compact metrizable space Δ_0 . Then the induced action of G on $\text{map}_{G_0}(G, \Delta_0) \cong \Delta_0^n$ is $n \cdot N\text{-}\mathcal{F}$ -amenable.

Proof. If E_0 is an (G_0, \mathcal{F}_0) -simplicial complex of dimension N_0 , then $E := \text{map}_{G_0}(G, E_0)$ is an (G, \mathcal{F}) -simplicial complex of dimension $n \cdot N_0$. The result follows from Lemma 4.3 and Lemma 4.2. \square

4.B. The Farrell Jones Conjecture. Let G be a group. Let \mathcal{A} be an additive category with a strict G -action and a strict direct sum. Following [5, Sec. 4.1] we will call such a category an additive G -category. For such a category \mathcal{A} there is then an additive category $\int_G \mathcal{A}$. If \mathcal{A} is equivalent to the category of finitely generated free R -modules and the G -action is trivial, then $\int_G \mathcal{A}$ is equivalent to the category of finitely generated free $R[G]$ -modules. Given a family \mathcal{F} of subgroups of G there is the K -theoretic \mathcal{F} -assembly map

$$\alpha_{G,\mathcal{F}}^K: H_*^G(E_{\mathcal{F}}G; \mathbf{K}_{\mathcal{A}}) \rightarrow K_*(\int_G \mathcal{A}).$$

The K -theoretic Farrell-Jones Conjecture (with coefficients) asserts that this map is an isomorphism if we use for \mathcal{F} the family VCyc of virtually cyclic subgroups [10, Conj. 3.2]. Farrell and Jones' original formulation [25] for the group ring $\mathbb{Z}[G]$ is a special case of this more general formulation.

If \mathcal{A} is in addition equipped with a strict involution [5, Sec. 4.1], then $\int_G \mathcal{A}$ inherits an involution and there is for any family \mathcal{F} of subgroups of G the L -theoretic \mathcal{F} -assembly map

$$\alpha_{G,\mathcal{F}}^L: H_*^G(E_{\mathcal{F}}G; \mathbf{L}_{\mathcal{A}}^{-\infty}) \rightarrow L_*^{\langle -\infty \rangle}(\int_G \mathcal{A}).$$

The L -theoretic Farrell-Jones Conjecture (with coefficients) asserts that this map is an isomorphism if we use for \mathcal{F} the family VCyc of virtually cyclic subgroups [4]. Again, Farrell and Jones' original formulation [25] for the group ring $\mathbb{Z}[G]$ is a special case of this more general formulation.

We will say that a group G satisfies the *Farrell-Jones Conjecture relative to \mathcal{F}* if for all G -additive categories \mathcal{A} (with strict involution in the L -theory case) the assembly maps $\alpha_{G,\mathcal{F}}^K$ and $\alpha_{G,\mathcal{F}}^L$ are isomorphisms. If $\mathcal{F} = \text{VCyc}$ then we will say that G satisfies the *Farrell-Jones Conjecture*. We will need the following results.

Theorem 4.6 (Transitivity principle). *Let \mathcal{F} be a family of subgroups of G . Assume that G satisfies the Farrell-Jones Conjecture relative to \mathcal{F} and that any subgroup $F \in \mathcal{F}$ satisfies the Farrell-Jones Conjecture. Then G satisfies the Farrell-Jones Conjecture.*

Proof. See for example [3, Thm. 2.10]. \square

Remark 4.7. An application of the Transitivity principle 4.6 is the following inheritance property for the Farrell-Jones conjecture for extensions, see for example [3, Thm. 2.7]. Let $N \rightarrow \hat{G} \rightarrow G$ be an extension. Suppose that G satisfies the Farrell-Jones Conjecture and that the preimage in \hat{G} of any virtually cyclic subgroup of G also satisfies the Farrell-Jones Conjecture. Then \hat{G} satisfies the Farrell-Jones Conjecture.

In the next result *ER* stands for Euclidean retract. Recall that a compact space X is a *Euclidean retract* (or *ER*) if it can be embedded in some \mathbb{R}^n as a retract. A compact metrizable space X is an *ER* if and only if it is a finite-dimensional contractible ANR.

Theorem 4.8. *Let \mathcal{F} be a family of subgroups of G that is closed under taking finite index overgroups. Suppose that G admits a finitely \mathcal{F} -amenable action on a compact *ER*. Then G satisfies the Farrell-Jones Conjecture relative to \mathcal{F} .*

Proof. This follows from the main axiomatic results of [5, 7]. The assumptions on G are formulated somewhat differently in these references, but it is not hard to check that the present assumptions imply the assumptions in these references, see also [2, Thm. 4.3]. \square

If F is a class of groups that is closed under taking subgroups and isomorphism, then for a group G we denote by $F(G)$ the family of subgroups of G that belong to F . For such a class of groups F we define the class $\mathbf{ac}(F)$ of groups to consist of all groups G that admit a finitely $F(G)$ -amenable action on a compact *ER*. Using the action on a point we see $F \subseteq \mathbf{ac}(F)$.

Lemma 4.9. *Let F be a class of groups that is closed under isomorphisms, taking subgroups, taking finite index overgroups, finite products and central extensions with finitely generated kernel. Then*

- (a) *if all groups in F satisfy the Farrell-Jones Conjecture, then all groups in $\mathbf{ac}(F)$ satisfy the Farrell-Jones Conjecture;*
- (b) *the class $\mathbf{ac}(F)$ is again closed under isomorphisms, taking subgroups, taking finite index overgroups, finite products and central extensions with finitely generated kernel.*

Proof. (a) By Theorem 4.8 every group G from $\mathbf{ac}(F)$ satisfies the Farrell-Jones Conjecture relative to $F(G)$. By assumption every group from $F(G)$ satisfies the Farrell-Jones Conjecture. Therefore the transitivity principle 4.6 implies that G satisfies the Farrell-Jones Conjecture.

(b) That $\mathbf{ac}(F)$ is closed under isomorphism and taking subgroups is

clear from the definition. Proposition 4.5 implies that it is also closed under finite index overgroups.

Let for $i = 1, 2$ the group G_i act finitely \mathcal{F}_i -amenable on Δ_i . Then the product action of $G_1 \times G_2$ on $\Delta_1 \times \Delta_2$ is finitely $\mathcal{F}_1 \times \mathcal{F}_2$ -amenable. Since \mathbf{F} is assumed to be closed under finite products it follows that $\mathbf{ac}(\mathbf{F})$ is also closed under finite products.

Let $C \rightarrow \hat{G} \rightarrow G$ be a central extension with C finitely generated. If G acts finitely \mathcal{F} -amenable on Δ , then \hat{G} acts via the projection $\hat{G} \rightarrow G$ finitely \mathcal{F}' -amenable on Δ , where \mathcal{F}' consists of central extensions with finitely generated kernel of groups in \mathcal{F} . Therefore, if $\mathcal{F} \subseteq \mathbf{F}$, then also $\mathcal{F}' \subseteq \mathbf{F}$. It follows that $\mathbf{ac}(\mathbf{F})$ is closed under central extensions with finitely generated kernel. \square

Starting with $\mathbf{ac}^0(\mathbf{F}) := \mathbf{F}$ we can define inductively $\mathbf{ac}^{n+1}(\mathbf{F}) := \mathbf{ac}(\mathbf{ac}^n(\mathbf{F}))$. We set $\mathbf{AC}(\mathbf{F}) := \bigcup \mathbf{ac}^n(\mathbf{F})$.

Corollary 4.10. *Let \mathbf{F} be a class of groups that is closed under isomorphisms, taking subgroups, taking finite index overgroups and finite products. Assume that all groups from \mathbf{F} satisfy the Farrell-Jones Conjecture. Then all groups from $\mathbf{AC}(\mathbf{F})$ satisfy the Farrell-Jones Conjecture.*

Proof. This follows by induction from Lemma 4.9. \square

Let \mathbf{VNil} be the class of finitely generated virtually nilpotent groups. Later, in Lemma 9.3, we will use Theorem B to show that mapping class groups of surfaces belong to $\mathbf{AC}(\mathbf{VNil})$. Thus, to prove Theorem A we will need the following well-known result.

Proposition 4.11. *All groups in the class \mathbf{VNil} satisfy the Farrell-Jones Conjecture.*

Proof. Finitely generated virtually abelian groups satisfy the Farrell-Jones Conjecture, see for example [3, Thm. 3.1].

Let $N \rightarrow \hat{G} \rightarrow G$ be an extension. If N is finitely generated and central, then all preimages of virtually cyclic groups are virtually finitely generated abelian. The inheritance property from Remark 4.7 now implies that the Farrell-Jones Conjecture is stable under central extensions with finitely generated kernel.

The case of finitely generated virtually nilpotent groups follows now by induction on the length of the lower central series. This induction is carried out in detail in an only marginally different situation in [8, Lem. 2.13]. \square

Remark 4.12. All virtually nilpotent subgroups of the mapping class group are known to be virtually abelian, see [16] and [38, Theorem

8.9]. Thus it may seem weird that nilpotent groups come up in our argument. Indeed, after a reorganization of the induction process we could avoid mentioning nilpotent groups, but we would still need the fact that central extensions (with finitely generated free abelian kernel) of groups satisfying the Farrell-Jones Conjecture satisfy the Farrell-Jones Conjecture. As explained above, the Farrell-Jones Conjecture for virtually nilpotent groups is an easy consequence of this fact and of the Farrell-Jones Conjecture for virtually abelian groups.

Remark 4.13. Lemma 4.9 (and its proof) remains true if we replace *central extension with finitely generated kernel* with *extension with abelian kernel*. Since Wegner [68] proved the Farrell-Jones Conjecture for the class \mathbf{VSol} of virtually solvable groups it follows that all groups in $\mathbf{AC}(\mathbf{VSol})$ satisfy the Farrell-Jones Conjecture as well. Wegner's proof is considerably more involved than the proof of the Farrell-Jones Conjecture for virtually finitely generated nilpotent groups given above.

It is not clear that $\mathbf{AC}(\mathbf{VSol})$ or $\mathbf{AC}(\mathbf{VNil})$ have all the inheritance properties known for the Farrell-Jones Conjecture. For example the Farrell-Jones Conjecture is also known to be stable under directed colimits and finite free products. If a group G acts finitely \mathcal{F} -amenably on a compact ER , then G is (strongly) transfer reducible relative to \mathcal{F} in the sense of [5, 67]. Groups that are transfer reducible relative to groups that satisfy the Farrell-Jones Conjecture satisfy the Farrell Jones Conjecture themselves. This is a consequence of the transitivity principle 4.6 and the main axiomatic results from [5, 67].

5. PRELIMINARIES ON MAPPING CLASS GROUPS

The general references for this section are [23, 26, 36]. Let Σ be a closed oriented surface of genus g , with $p \geq 0$ punctures (i.e. distinguished points). We denote by $P \subset \Sigma$ the set of punctures. The *mapping class group*

$$\mathrm{Mod}(\Sigma) = \pi_0(\mathrm{Homeo}_+(\Sigma, P))$$

is the group of components of the group of orientation-preserving homeomorphisms of Σ that leave P invariant.

We will always assume $6g + 2p - 6 > 0$ regarding the other cases as sporadic. This condition is equivalent to the existence of more than one complete hyperbolic structure of finite area on $\Sigma \setminus P$.

The sporadic cases are $g = 0$, $p \leq 3$ when $\mathrm{Mod}(\Sigma)$ is finite, and $g = 1$, $p = 0$ when $\mathrm{Mod}(\Sigma) = \mathrm{SL}_2(\mathbb{Z})$ is virtually free.

A simple closed curve in $\Sigma \setminus P$ is *essential* if it does not bound a disk or a once punctured disk. We denote by \mathcal{S} the set of isotopy classes

of essential simple closed curves in $\Sigma \setminus P$ and refer to its elements as *curves*. If $\Sigma \setminus P$ is given a complete hyperbolic structure of finite area, every $s \in \mathcal{S}$ has a unique geodesic representative. If $s, s' \in \mathcal{S}$ the *intersection number* $i(s, s')$ is the smallest cardinality of $a \cap a'$ as a, a' range over simple closed curves in the isotopy classes s, s' respectively. Thus $i(s, s) = 0$ and for $s \neq s'$ $i(s, s')$ is the cardinality of the intersection between the geodesic representatives of s, s' . See [23, Section 1.2] or [19, Lemma 2.6].

To Σ one associates several spaces on which the mapping class group $\text{Mod}(\Sigma)$ acts.

5.A. Teichmüller space. The *Teichmüller space* $\mathcal{T} = \mathcal{T}(\Sigma)$ is the space of marked complex structures on Σ with P the set of distinguished points. Equivalently, by the Uniformization Theorem, \mathcal{T} is the space of marked complete hyperbolic structures of finite area on $\Sigma \setminus P$. Then \mathcal{T} is naturally a smooth (or even complex analytic) manifold diffeomorphic to $\mathbb{R}^{6g+2p-6}$. The mapping class group $\text{Mod}(\Sigma)$ acts by changing the marking. This action is discrete but not cocompact; however, there are natural cocompact subspaces. Fix an $\epsilon > 0$ and consider the *thick part* $\mathcal{T}_{\geq \epsilon} \subset \mathcal{T}$ consisting of $X \in \mathcal{T}$ such that every closed hyperbolic geodesic has length $\geq \epsilon$.

Theorem 5.1 (Mumford [58]). *The thick part $\mathcal{T}_{\geq \epsilon} \subset \mathcal{T}$ is cocompact.*

Thus for $\epsilon_n \searrow 0$, the sequence $\mathcal{T}_{\geq \epsilon_n}$ forms an exhaustion of \mathcal{T} by cocompact subsets.

For $X \in \mathcal{T}$ we use the marking to identify the set of curves on X with \mathcal{S} .

5.B. Measured foliations. This is an important tool introduced by Thurston, see [26]. It provides a “completion” of the set \mathcal{S} , much like the circle is a completion of $\mathbb{Q} \cup \{\infty\}$.

A *measured foliation* on Σ is a foliation with finitely many singularities equipped with a transverse measure of full support. The singularities are standard k -prong singularities with $k \geq 3$ except that $k = 1$ is allowed at the punctures. Each leaf is either an arc joining two singularities or punctures (that may coincide), or an essential circle, or an injectively immersed line or ray that starts at a puncture or a singular point.

The *Whitehead equivalence* is generated by collapsing leaves that are arcs joining distinct singularities and isotopies. Every measured foliation μ determines a length function $\ell_\mu : \mathcal{S} \rightarrow [0, \infty)$ that sends $s \in \mathcal{S}$ to the infimum of measures over simple closed curves in the isotopy class s .

Every curve $s \in \mathcal{S}$ also determines a length function $\ell_s : \mathcal{S} \rightarrow [0, \infty)$ via $\ell_s(s') = i(s, s')$. There is a unique equivalence class $j(s)$ of measured foliations such that $\ell_{j(s)} = \ell_s$. The function $j : \mathcal{S} \hookrightarrow \mathcal{MF}$ is the *canonical inclusion*. The foliation $j(s)$ has all but finitely many leaves in the isotopy class s . We will usually suppress j and write $\mathcal{S} \subset \mathcal{MF}$.

The set of all measured foliations up to Whitehead equivalence has a natural topology, homeomorphic to $\mathbb{R}^{6g+2p-6} - \{0\}$. It is defined by embedding the set of measured foliations in the space of length functions $\ell : \mathcal{S} \rightarrow [0, \infty)$. Adding the “empty foliation” 0, one obtains the space \mathcal{MF} homeomorphic to $\mathbb{R}^{6g+2p-6}$, and projectivizing with respect to the action of \mathbb{R}_+ that scales the measure, the space \mathcal{PMF} homeomorphic to $S^{6g+2p-7}$. The subset $\mathcal{S} \subset \mathcal{MF}$ is closed and discrete, but after projectivizing, the image of \mathcal{S} in \mathcal{PMF} is dense. The intersection pairing on \mathcal{S} extends uniquely to $i : \mathcal{MF} \times \mathcal{MF} \rightarrow [0, \infty)$ in such a way that it is continuous and \mathbb{R}_+ -equivariant in each variable. Moreover, i is symmetric and $i(\mu, s) = \ell_\mu(s)$ when $s \in \mathcal{S}$, and $i(\mu, \mu) = 0$ for every $\mu \in \mathcal{PMF}$. The intersection pairing does not descend to \mathcal{PMF} ; however the statement $i(\xi, \eta) = 0$ (or $\neq 0$) makes sense for projectivized measured foliations ξ, η .

A measured foliation μ is *filling* if $i(\mu, s) > 0$ for every $s \in \mathcal{S}$. This is equivalent to the condition that no curve $s \in \mathcal{S}$ can be homotoped into the union of finitely many leaves. If μ is filling, the set $\Delta(\mu) = \{[\nu] \in \mathcal{PMF} \mid i(\mu, \nu) = 0\}$ has the structure of a simplex [43, Theorem 14.7.6] and consists of classes of measures with the same underlying foliation as μ (for the latter see [64, Theorem 1.12]). The vertices correspond to ergodic measures, and general points to convex combinations of ergodic measures. One characterization of an ergodic measure μ is that if it is written as the sum $\mu = \mu_1 + \mu_2$ of transverse measures then necessarily both μ_i are multiples of μ . When the simplex degenerates to a point, μ is called *uniquely ergodic*.

Thurston [26] constructed an equivariant compactification $\overline{\mathcal{T}}$ of \mathcal{T} such that $\overline{\mathcal{T}} - \mathcal{T} = \mathcal{PMF}$ and the pair $(\overline{\mathcal{T}}, \mathcal{T})$ is homeomorphic to the pair $(B, \text{int}B)$ where B is the closed ball of dimension $6g + 2p - 6$. This compactification can be described as follows. A point $X \in \mathcal{T}$, thought of as a hyperbolic surface, determines a length function ℓ_X by sending $s \in \mathcal{S}$ to its hyperbolic length. Then a sequence of hyperbolic surfaces converges to the projective class of a measured foliation if the corresponding length functions converge projectively to the length function of the foliation.

5.C. Measured geodesic laminations. A *geodesic lamination* in a complete hyperbolic surface Σ of finite area is a nonempty compact

subset of $\Sigma \setminus P$ which is a disjoint union of geodesics. A *measured geodesic lamination* is a geodesic lamination equipped with a transverse measure. There is a natural bijection between the set \mathcal{MF} of measured foliations up to isotopy and Whitehead moves, and the set \mathcal{ML} of measured geodesic laminations. See [47] and [41, Chapter 11]. If a measured foliation μ corresponds to the measured lamination ξ then $i(s, \mu) = i(s, \xi)$ for every $s \in \mathcal{S}$. A rough description of the correspondence is as follows. Let ℓ be a generic leaf of μ . Its lift to the universal cover of $\Sigma \setminus P$ (which can be identified with hyperbolic plane via a complete finite area hyperbolic metric on $\Sigma \setminus P$) is a quasi-geodesic which is bounded distance away from a unique infinite geodesic l . The image of l is a generic leaf of ξ .

When ξ is a measured geodesic lamination, denote by $|\xi|$ its support, i.e. the union of those leaves of ξ such that the measure of any arc crossing it transversally is nonzero.

5.D. The supporting multisurface. Consider a measured geodesic lamination ξ . The support $|\xi|$ is a geodesic lamination with finitely many components and each is minimal (i.e. every leaf is dense), including the possibility of a simple closed geodesic. Since we require that $|\xi|$ be compact, there are no leaves going to punctures. Even more generally, a geodesic lamination (possibly not the support of a measure) consists of finitely many minimal components and finitely many isolated leaves, each of which is either closed or in each direction spirals towards a closed leaf or a minimal component. See e.g. [17, Proposition 3]. The spiraling leaves cannot be in the support of a measure since they would give rise to transverse arcs with infinite measure.

We say that ξ or $|\xi|$ is *filling* if every complementary component of $|\xi|$ is homeomorphic to an open disk or to an open once punctured disk. Equivalently, every simple closed geodesic α in Σ intersects $|\xi|$, or equivalently again, $i(\alpha, \xi) > 0$, i.e. the corresponding measured foliation is filling. If ξ is filling then $|\xi|$ is connected.

Unless otherwise stated, when we talk about subsurfaces $Y \subset \Sigma$ we mean

- connected and closed, as subsets of Σ ,
- no punctures on the boundary,
- $Y \neq \Sigma$,
- Y is not a disk or a once punctured disk or a pair of pants, by which we mean a sphere with the total of exactly three punctures and boundary components,
- no complementary component is a disk or a punctured disk,
- up to isotopy rel P .

In particular, subsurfaces are π_1 -injective.

A *multisurface* is a nonempty disjoint union of subsurfaces that does not contain distinct annuli which are isotopic rel P .

When $|\xi|$ is connected but not filling there is a unique subsurface $Y \subset \Sigma$ that contains $|\xi|$ and ξ is filling in Y . We call Y the *supporting subsurface* of ξ and denote it $Supp(\xi)$ or $Supp(|\xi|)$. That ξ fills $Supp(|\xi|)$ means that $i(s, \xi) > 0$ for every essential curve s in $Supp(|\xi|)$ not homotopic into $\partial(Supp(|\xi|))$.

We note that the supporting subsurface of a simple closed curve is an annulus, and otherwise the supporting subsurface has negative Euler characteristic and cannot be a pair of pants (from the point of view of foliations this was proved in [26, Exposé 6]).

In general, when $|\xi|$ is disconnected, the supporting subsurfaces of the components can be isotoped so that they are pairwise disjoint. The union of these supporting subsurfaces of the components is by definition the supporting multisurface $Supp(\xi)$ or $Supp(|\xi|)$.

Now that we made the careful distinction, we will revert to the standard terminology and call $Supp(\xi)$ the supporting subsurface even when it is not connected.

The set of geodesic laminations in Σ is a compact space with respect to Hausdorff topology on the space of compact subsets of Σ . The following is standard.

Proposition 5.2. *Suppose $\xi_n \rightarrow \xi$ is a convergent sequence of measured geodesic laminations, and suppose that $|\xi_n| \rightarrow \lambda$ in the Hausdorff topology. Then $|\xi| \subseteq \lambda$.*

Proof. Let s be a curve in the complement of $Supp(\lambda)$. Then $i(s, \xi_n) = 0$ for large n since s is disjoint from $Supp(\xi_n)$. It follows that $i(s, \xi) = 0$, so s is disjoint from the support of ξ . \square

5.E. The Teichmüller metric. The Teichmüller space \mathcal{T} is equipped with a proper geodesic metric which is $\text{Mod}(\Sigma)$ -invariant. The distance between two complex surfaces is defined to be

$$d_{\mathcal{T}}(X, Y) = \inf \log(K_f)$$

where f ranges over all orientation preserving homeomorphisms $X \rightarrow Y$ which are smooth except at finitely many points, and

$$K_f = \sup K_f(p)$$

is the supremum of dilatations $K_f(p)$ over the points $p \in X$ where f is smooth (it is customary to scale this expression by $\frac{1}{2}$ but we will ignore this). Recall that $K_f(p) \geq 1$ is the ratio of major to minor axes of the ellipse obtained by taking a round circle and applying the derivative

df_p . Teichmüller proved that the infimum of K_f is realized by a unique homeomorphism, called the *Teichmüller map*. The Teichmüller metric is proper and $\text{Mod}(\Sigma)$ -invariant.

5.F. Holomorphic quadratic differentials. The cotangent space of \mathcal{T} at $X \in \mathcal{T}$ is the space of (holomorphic) quadratic differentials on X , each of which is defined in charts as $q(z) = f(z)dz^2$ with f holomorphic, possibly with simple poles at the punctures. See e.g. [37] for basic facts about quadratic differentials.

A nonzero quadratic differential q on $X \in \mathcal{T}$ determines two measured foliations, horizontal q^H and vertical q^V . Away from the singularities, i.e. points where q has a zero or a pole, there are charts where $q = dz^2$, and then the vertical foliation is defined by the vertical lines and with transverse measure $|dx|$, and similarly for the horizontal foliation. The quadratic differential q also determines a Euclidean metric on Σ with cone type singularities: on a chart where $q = dz^2$ the metric is Euclidean.

The norm of q is $\|q\| = \int_X |q|$, i.e. it equals the area of X with respect to the Euclidean metric. We denote by $QD(X)$ the vector space of all quadratic differentials on X and by $QD^1(X)$ the subset of unit norm quadratic differentials.

Geodesics in \mathcal{T} (i.e. *Teichmüller geodesics*) have a simple description in terms of quadratic differentials. If $X \in \mathcal{T}$ and q is a unit norm quadratic differential on X , for $t \in \mathbb{R}$ define $X_t \in \mathcal{T}$ by the rule that on a chart of X where $q = dz^2$, the chart for X_t is $x + iy \mapsto e^{t/2}x + ie^{-t/2}y$. Then $t \mapsto X_t$ is a geodesic line determined by X and q and it is parametrized with unit speed. The identity map $X \rightarrow X_t$ is the Teichmüller map for these two points in \mathcal{T} . There is a natural quadratic differential q_t on X_t given by $q_t = dz^2$ in the new charts, and the Teichmüller geodesic defined by (X_{t_0}, q_{t_0}) is the same as the one defined by (X, q) except for the reparametrization $t \mapsto t + t_0$. By definition we have

$$q_t^H = e^{t/2}q^H \quad \text{and} \quad q_t^V = e^{-t/2}q^V.$$

We have a map from the cone

$$\widehat{QD}^1(X) = QD^1(X) \times [0, \infty) / QD^1(X) \times \{0\}$$

to \mathcal{T} given by $(q, t) \mapsto X_t$ with X_t described above.

Theorem 5.3 (Teichmüller's contractibility theorem). *This map is a homeomorphism.*

For a proof see e.g. [36, Theorem 7.2.1]. A consequence of the theorem is that any two points in \mathcal{T} are joined by a unique Teichmüller geodesic.

6. PROJECTIONS

We start with background.

6.A. Curve complex; arc and curve complex. The *curve complex* $\mathcal{C}(\Sigma)$ is the simplicial complex whose vertex set is the set of curves in \mathcal{S} , and simplices are collections of curves that have pairwise disjoint representatives. When $6g - 6 + 2p > 2$ this complex is connected.

The following is a celebrated theorem of Masur and Minsky.

Theorem 6.1 ([54]). *The 1-skeleton of $\mathcal{C}(\Sigma)$ is a δ -hyperbolic graph whenever it is connected.*

It will be more convenient to work with the *arc and curve complex* $\mathcal{AC}(\Sigma)$. Its vertices are represented by (essential) arcs and curves. By an *arc* we mean a path in Σ whose interior points are in $\Sigma \setminus P$ and whose boundary is in P , and it is embedded except possibly at the endpoints. Two arcs are equivalent if they are homotopic through arcs. An arc is essential if it is not homotopic through arcs to a small neighborhood of a puncture. A simplex in $\mathcal{AC}(\Sigma)$ is a collection of arcs and curves that have disjoint representatives, except possibly for the endpoints of arcs.

The complex $\mathcal{AC}(\Sigma)$ is connected and δ -hyperbolic as soon as $6g - 6 + 2p > 0$. When $6g - 6 + 2p > 2$ the natural inclusion

$$\mathcal{C}(\Sigma) \hookrightarrow \mathcal{AC}(\Sigma)$$

is a quasi-isometry. The inverse is constructed by sending an arc α to an essential component of the boundary of the regular neighborhood of α , see [45, Theorem 1.3].

When $6g - 6 + 2p = 2$ (i.e. when (Σ, P) is the once-punctured torus or the four times punctured sphere) the complex $\mathcal{AC}(\Sigma)$ is quasiisometric to the Farey graph (hence also hyperbolic), while $\mathcal{C}(\Sigma)$ is an infinite discrete space.

If α, β are two arcs or curves, their intersection number $i(\alpha, \beta)$ is the smallest cardinality of the intersection of their representatives, not counting the punctures.

We have the following useful estimate on the distance in $\mathcal{C}(\Sigma)$ and $\mathcal{AC}(\Sigma)$.

Proposition 6.2. • *If α, β are curves and $6g - 6 + 2p > 2$ then $d_{\mathcal{C}(\Sigma)}(\alpha, \beta) \leq i(\alpha, \beta) + 1$.*

• *If α, β are arcs or curves and $P \neq \emptyset$, then $d_{\mathcal{AC}(\Sigma)}(\alpha, \beta) \leq i(\alpha, \beta) + 3$.*

Proof. The first claim is well known; it can be easily proved by induction on the intersection number using surgery. There are also logarithmic bounds, see [33].

The second claim can be proved similarly. E.g. see [34] for the case when α, β are arcs. If α is a curve we can construct an arc α' disjoint from α and with $i(\alpha', \beta) \leq i(\alpha, \beta)$. If β is also a curve we can similarly replace it with an arc β' and then we have $d_{\mathcal{AC}(\Sigma)}(\alpha, \beta) \leq d_{\mathcal{AC}(\Sigma)}(\alpha', \beta') + 2 \leq i(\alpha, \beta) + 3$. \square

When Σ is a 3 times punctured sphere, the complex $\mathcal{AC}(\Sigma)$ is finite, and is not useful when considering subsurface projections.

6.B. Curve complex of the annulus. When A is an annulus, we define $\mathcal{C}(A) = \mathcal{AC}(A)$ to be the graph whose vertices are embedded arcs with endpoints on distinct boundary components of A , modulo isotopy rel boundary, and edges correspond to disjointness. Thus $\mathcal{C}(A)$ is quasi-isometric to \mathbb{Z} .

6.C. The Gromov boundary. Klarreich [44] identified the Gromov boundary of the curve complex $\mathcal{C}(\Sigma)$ (or equivalently of $\mathcal{AC}(\Sigma)$). A point in $\partial\mathcal{C}(\Sigma)$ is represented by a filling measured geodesic lamination ξ and two such laminations ξ, ξ' represent the same point if $|\xi| = |\xi'|$ (see Section 5.C). In other words, a point in $\partial\mathcal{C}(\Sigma)$ is a filling geodesic lamination that admits a transverse measure of full support.

We now state Klarreich's work in more detail. First recall that if x_n is a sequence in a δ -hyperbolic space X then after a subsequence one of the following occurs:

- $x_n \rightarrow z \in \partial X$, or
- there is some $x \in X$ so that x_n *coarsely rotates* around x . This means that for any n there is m_0 so that for $m > m_0$ any geodesic $[x_n, x_m]$ passes within a uniform distance (e.g. 10δ) from x .

This statement is really an exercise in Gromov products (e.g. the reader should contemplate the case of a locally infinite tree). A more sophisticated approach is via the horofunction boundary, see e.g. [50, Section 3].

The theorem of Klarreich can now be summarized as follows.

Theorem 6.3 (Klarreich). *There is a coarse map $\pi : \overline{\mathcal{T}} \rightarrow \mathcal{C}(\Sigma) \cup \partial\mathcal{C}(\Sigma)$ with the following properties.*

- (1) *Suppose $x_n \in \overline{\mathcal{T}}$, $x_n \rightarrow x \in \overline{\mathcal{T}}$. If $\pi(x) \in \partial\mathcal{C}(\Sigma)$ then $\pi(x_n) \rightarrow \pi(x)$. If $\pi(x) \in \mathcal{C}(\Sigma)$ then $\pi(x_n)$ coarsely rotates around $\pi(x)$.*
- (2) *If $X \in \mathcal{T}$ then $\pi(X)$ is the collection of shortest curves on X (or equivalently, collection of curves of length less than a suitable*

constant). If $\mu \in \mathcal{PMF}$ is not filling, $\pi(\mu)$ consists of boundary components of the supporting multisurface. If μ is filling then $\pi(\mu) \in \partial\mathcal{C}(\Sigma)$. Moreover, for every $b \in \partial\mathcal{C}(\Sigma)$ the preimage $\pi^{-1}(b)$ is nonempty and consists of the simplex of projectivized transverse measures on the same underlying foliation. In particular, if μ is uniquely ergodic, the preimage of $\pi(\mu)$ is a single point.

If A is an annulus, its curve complex is quasi-isometric to \mathbb{Z} and the Gromov boundary has two points. We can think of them as the two ways in which a geodesic can spiral in and out of the annulus represented by a regular neighborhood of a simple closed geodesic.

6.D. Subsurface projections. This key concept was introduced by Masur and Minsky [55]. Recall our convention about subsurfaces from Section 5.D. In particular, they are proper and connected. When a subsurface Y is not an annulus, we define its arc and curve complex $\mathcal{AC}(Y)$ as $\mathcal{AC}(\hat{Y})$, where \hat{Y} is obtained from Y by collapsing each boundary component to a puncture. Thus any essential arc with boundary in ∂Y represents a point in $\mathcal{AC}(Y)$. The complex $\mathcal{AC}(Y)$ is always δ -hyperbolic and of infinite diameter since Y is not allowed to be a pair of pants.

Let $Y \subset \Sigma$ be a connected subsurface different from a pair of pants. Fix a complete hyperbolic metric of finite area on $\Sigma \setminus P$ and realize all nonperipheral boundary components of Y by geodesics. If no two boundary components of Y are parallel, then Y is realized as a totally geodesic subsurface.

Let α be an arc or a curve in Σ , not isotopic into the complement of Y , realized as a geodesic. The intersection $Y \cap \alpha$ is a curve or a collection of arcs. We define $\pi_Y(\alpha) \subset \mathcal{AC}(Y)$ to be this intersection. This is a collection of points in $\mathcal{AC}(Y)$ at pairwise distance ≤ 1 , so coarsely the projection is well-defined.

If Y has parallel boundary components but is not an annulus (i.e. when a complementary component is an annulus) consider the covering space $\Sigma_Y \rightarrow \Sigma$ corresponding to $Y \subset \Sigma$. The subsurface Y lifts to Σ_Y and there is a unique representative, up to isotopy, which is totally geodesic, and we will identify it with Y . The entire covering space Σ_Y is obtained from Y by attaching half-open annuli to the boundary components. Each annulus is of the form H/\mathbb{Z} , where H is the hyperbolic half-plane and \mathbb{Z} acts by translation along the boundary. The Gromov compactification of H/\mathbb{Z} is a (compact) annulus, and attaching these annuli to Y produces a surface $\bar{\Sigma}_Y$ homeomorphic to Y , with homeomorphism being canonical up to isotopy. Now define $\pi_Y(\alpha) \subset \mathcal{AC}(Y)$

as the intersection of the preimage of α in Σ_Y with Y . Equivalently, identifying Y with $\overline{\Sigma}_Y$, take the closure of the preimage of α , and discard the inessential components. The resulting finite collection of arcs (or a curve) is the projection.

When Y is the annulus, it is the latter description of the projection that generalizes. Namely, $\overline{\Sigma}_Y$ is an annulus. Again take the closure of the preimage of α , and discard the inessential components to get the projection.

We make the same definition when α is a collection of pairwise disjoint arcs or curves and at least one is not isotopic into the complement of Y .

Subsurface projections can also be defined for other subsurfaces and for geodesic laminations.

If $Y' \subset \Sigma$ is another subsurface, define

$$\pi_Y(Y') = \pi_Y(\partial Y')$$

assuming the latter is defined; otherwise $\pi_Y(Y')$ is undefined.

6.E. Projecting geodesic laminations. We now define $\pi_Y(\xi) = \pi_Y(|\xi|)$ when ξ is a measured geodesic lamination with support $|\xi|$. As suggested by the notation, it will depend only on the support, and it will be defined whenever $Y \cap \text{Supp}(\xi) \neq \emptyset$ (even after isotopy). If Y is a component of $\text{Supp}(\xi)$ and it is not an annulus we define $\pi_Y(\xi)$ to be the point at infinity of $\mathcal{AC}(\Sigma)$ represented by $|\xi|$.

If Y is an annulus component of $\text{Supp}(\xi)$ we define $\pi_Y(\xi)$ to be the two points at infinity in the curve complex.

Now suppose that Y is not a component of $\text{Supp}(\xi)$. First assume that Y is realized as a totally geodesic subsurface of Σ . Then $|\xi| \cap Y$ is the union of a collection of arcs (typically uncountably many, but there are only finitely many isotopy classes) and the underlying set of a measured geodesic lamination μ . Define $\pi_Y(\xi)$ as the set of these arcs and boundary components of $\text{Supp}(\mu)$ that are not boundary components of Y .

More generally, if Y is not an annulus, we lift to the cover Σ_Y and intersect with the totally geodesic copy of Y .

Finally, if Y is an annulus, it is crossed by some leaves of ξ . Lift those leaves to Σ_Y and take their closure in $\overline{\Sigma}_Y$ to get $\pi_Y(\xi)$.

Note that the set $\mathcal{PMF}(Y) \subset \mathcal{PMF}$ consisting of measured geodesic laminations ξ such that $\pi_Y(\xi)$ is defined is open. This follows from Proposition 5.2.

6.F. Projection distance. Let α, β be two curves, or arcs, or subsurfaces, or Riemann surfaces, or measured foliations, so that the projections $\pi_Y(\alpha)$ and $\pi_Y(\beta)$ to a subsurface Y are defined. Then define the *projection distance*

$$d_Y^\pi(\alpha, \beta) = \text{diam}(\pi_Y(\alpha) \cup \pi_Y(\beta))$$

When we use this notation at most one of $\pi_Y(\alpha), \pi_Y(\beta)$ will represent a point (or points) at infinity, and in this case the projection distance is infinite. In all other cases, since the diameter of $\pi_Y(\alpha)$ is uniformly bounded, the projection distance is a well-defined finite number.

The following triangle inequality is obvious.

Proposition 6.4. *If $\pi_Y(\alpha), \pi_Y(\beta), \pi_Y(\gamma)$ are all defined, then*

$$d_Y^\pi(\alpha, \beta) + d_Y^\pi(\beta, \gamma) \geq d_Y^\pi(\alpha, \gamma)$$

The following key inequality was proved by Behrstock. We say that two subsurfaces Y, Y' *overlap* if $\partial Y \cap \partial Y' \neq \emptyset$.

Proposition 6.5 ([11]). *There is a constant C such that the following holds. Let Y, Z be two overlapping subsurfaces of X and let α be a collection of pairwise disjoint arcs and curves. Assume $\pi_Y(\alpha)$ and $\pi_Z(\alpha)$ are defined. Then*

$$d_Y^\pi(\alpha, \partial Z) \geq C \implies d_Z^\pi(\alpha, \partial Y) \leq C$$

The same statement holds when α is replaced by a foliation.

Leininger supplied explicit constant $C = 10$ and a simple argument, see [51, Lemma 2.13]. When α is replaced by a foliation ξ the proof is an easy consequence, as follows:

- Leininger's proof works with no change if some leaf of ξ intersects ∂Y or ∂Z (in particular, this occurs if ξ is filling).
- If ξ is not filling let $\alpha = \partial \text{Supp}(\xi)$. If $\pi_Y(\alpha)$ and $\pi_Z(\alpha)$ are defined, the claim about ξ follows (after increasing C by 1) after observing that $d_Y(\alpha, \xi) \leq 1$ and $d_Z(\alpha, \xi) \leq 1$.
- If Y is a component of $\text{Supp}(\xi)$, then $d_Z^\pi(\xi, \partial Y) \leq 1$, and similarly for Z .

The following was first proved in [55]. A streamlined proof with the explicit bound is in [12, Lemma 5.3].

Proposition 6.6. *For any two subsurfaces Y, Z there are only finitely many subsurfaces W such that $d_W^\pi(Y, Z) > 3$.*

6.G. The Bounded Geodesic Image Theorem. This theorem was proved by Masur and Minsky [55]. A more combinatorial proof with a uniform bound on M was given by Webb [66].

Theorem 6.7. *There exists $M = M(\Sigma)$ such that the following holds. Let $Y \subset \Sigma$ be a subsurface and $g = x_0, x_1, \dots, x_n$ a geodesic in $\mathcal{C}(\Sigma)$ such that $\pi_Y(x_i)$ is defined for all i . Then $d_Y^\pi(x_0, x_n) \leq M$.*

Proposition 6.8. *There is a constant $N = N(\Sigma)$ so that the following holds. Suppose $\pi_Y(\nu)$ is defined and let $\alpha \in \mathcal{AC}(Y)$ and $\Theta \in [0, \infty)$ such that $d_Y^\pi(\alpha, \nu) \geq \Theta$. Then there is a neighborhood U of ν in \mathcal{PMF} such that $\pi_Y(\mu)$ is defined and in addition*

$$d_Y^\pi(\alpha, \mu) > \Theta - N$$

for all $\mu \in U$.

Proof. We already noted that π_Y is defined on an open subset of \mathcal{PMF} . Let $\mu_i \rightarrow \nu$ in \mathcal{PMF} .

We first consider the case when Y is a component of $\text{Supp}(\nu)$. If Y is also a component of $\text{Supp}(\mu_i)$ there is nothing to prove. Otherwise, the projection $\pi_Y(\mu_i)$ consists of a collection of curves (coming from boundary components of the support of μ_i) and of a collection of arcs. Denote by λ_i either one of the curves in the collection, or one of the essential nonperipheral boundary components of $I \cup \partial Y$ where I is one of the arcs in the collection. We view λ_i as a measured lamination, and we may assume $\lambda_i \rightarrow \lambda \in \mathcal{PMF}$. Then λ is supported in Y and satisfies $i(\lambda, \nu) = 0$. Since $\pi_Y(\nu)$ fills Y it follows that $|\lambda| = |\pi_Y(\nu)|$. By Theorem 6.3 it follows that $d_Y^\pi(\alpha, \lambda_i) \rightarrow \infty$, so we are done in this case since $d_Y^\pi(\lambda_i, \mu_i) \leq 5$, see Proposition 6.2. When Y is an annulus we cannot use Theorem 6.3, but argue directly as follows. Let α be the closed geodesic homotopic into Y . If μ_i has α as a closed leaf then $d_Y(\alpha, \mu_i) = \infty$ so we are done. Otherwise, μ_i has leaves that intersect α at a small angle by Proposition 5.2 and then again the projections go to infinity.

Second, consider the case when a leaf ℓ of $|\nu|$ intersects ∂Y . By Proposition 5.2 there is a leaf ℓ_i of μ_i for large i that also intersects ∂Y and an arc component of $\ell_i \cap Y$ is isotopic to an arc component of $\ell \cap Y$. Thus in this case $d_Y^\pi(\mu_i, \nu) \leq 2$ (if Y is an annulus we only get $d_Y^\pi(\mu_i, \nu) \leq 3$), so here $N > 3$ works.

Finally, suppose that a component Z of $\text{Supp}(\nu)$ is isotopic into Y , but is not Y . By the same argument as in the first case, we see that $\pi_Z(\mu_i)$ go to infinity in $\mathcal{AC}(Z)$ (again argue directly if Z is an annulus). After passing to a subsequence, we may assume that for $i < j$ we have that $d_Z^\pi(\mu_i, \mu_j)$ is large. By the Bounded Geodesic Image Theorem 6.7,

we see that any geodesic joining $\pi_Y(\mu_i)$ and $\pi_Y(\mu_j)$ must contain a curve disjoint from Z , and so is within distance 1 from $\pi_Y(\nu)$. Then by δ -hyperbolic geometry we deduce that $d_Y^\pi(\alpha, \nu) \leq \Theta - 1 - \delta$ for at most one i . So $N > 1 + \delta$ works in this case. \square

6.H. Partitioning the subsurfaces and the color preserving subgroup. We will need the following fact.

Proposition 6.9 ([12, Proposition 5.8]). *The set of subsurfaces of Σ which are not pairs of pants can be written as a finite disjoint union*

$$\mathbf{Y}^1 \sqcup \mathbf{Y}^2 \sqcup \dots \sqcup \mathbf{Y}^k$$

so that any two subsurfaces in any \mathbf{Y}^i overlap, and there is a subgroup $G < \text{Mod}(\Sigma)$ of finite index that preserves each \mathbf{Y}^i .

The subgroup G is the *color preserving subgroup*. We can further arrange that each \mathbf{Y}^i is a G -orbit. We remark that unlike in [12], $\{\Sigma\}$ is not one of the \mathbf{Y}^i since we are considering only proper subsurfaces.

6.I. Large intersection number implies large projection. Notice that one can have two curves with large intersection number that are at distance 2 in the curve complex. Thus the literal converse to Proposition 6.2 does not hold. The following is the correct converse to Proposition 6.2.

Lemma 6.10. [20, 65] *For every M there is I such that the following holds. If $\alpha, \beta \in \mathcal{S}$ and $i(\alpha, \beta) \geq I$ then there is a subsurface $Y \subset \Sigma$ such that $d_Y(\alpha, \beta) \geq M$ (where we also allow $Y = \Sigma$).*

The same statement holds for pairs of arcs-or-curves in $\mathcal{AC}(\Sigma)$.

7. VERIFICATION OF THE FLOW AXIOMS

Let c be a geodesic (segment, ray or line) in a metric space X . The *nearest point projection* is the function ρ_c that to a point $x \in X$ assigns the subset of $\text{Im}(c)$ consisting of the minima of the proper map $\text{Im}(c) \rightarrow [0, \infty)$ defined by $z \mapsto d(x, z)$.

Definition 7.1. The geodesic c is *D -contracting* for $D \geq 0$ if for every metric ball $B \subset X$ with $B \cap \text{Im}(c) = \emptyset$ the set

$$\rho_c(B) := \cup_{b \in B} \rho_c(b) \subset \text{Im}(c)$$

has diameter $\leq D$.

For example, if X is δ -hyperbolic then every geodesic is 10δ -contracting. It is an interesting phenomenon that many spaces of interest, even though they are not hyperbolic, contain many contracting

geodesics. These are thought of as “hyperbolic directions”. Note that a line in \mathbb{R}^2 is not contracting.

Definition 7.2. A geodesic (segment, ray or line) c in a metric space is *Morse* if for every A, B there is $N = N(A, B)$ so that any (A, B) -quasi-geodesic c' with endpoints in c is contained in the N -neighborhood $B_N(c)$ of c .

The following lemma is well known. The first part states that contracting geodesics are Morse, and the second is a variant for geodesic lines.

Lemma 7.3. (i) *For every D, A, B there is $N = N(D, A, B)$ such that every (A, B) -quasi-geodesic with endpoints on a D -contracting geodesic c is contained in the N -neighborhood of c .*
(ii) *Suppose c is a contracting geodesic line and c' is a geodesic line contained in some neighborhood $B_M(c)$ of c . Then c' is contained in the R -neighborhood of c , where R is a function of the contracting constant D .*

For the first statement, see e.g. [1]. The idea is that if c' contains a long segment outside a big neighborhood of c , then covering this segment with slightly overlapping big balls that miss c and projecting we see that the projection of the segment to c has much smaller length, and this contradicts the assumption that c' is a quasi-geodesic. The second statement can be proved similarly.

We will need the following facts about Teichmüller geodesics. Recall that $\pi : \mathcal{T} \rightarrow \mathcal{C}(\Sigma)$ is the coarse projection to the curve complex.

Proposition 7.4. (1) *(Arzela-Ascoli) Any sequence of Teichmüller geodesics that intersect a fixed compact set has a subsequence that (after reparametrization via translation) converges to a Teichmüller geodesic.*
(2) *If c is a Teichmüller geodesic (segment, ray or line) in the thick part $\mathcal{T}_{\geq \epsilon}$ then*

- *πc is a (K, L) -quasi-geodesic in $\mathcal{C}(\Sigma)$, where K, L depend only on ϵ . For a much more general statement see [61].*
- *(Minsky [57]) c is D -contracting for $D = D(\epsilon)$.*
- *(the Masur Criterion [53]) $c(t)$ converges as $t \rightarrow \infty$ to $c(\infty) \in \mathcal{PMF}$ which is filling and uniquely ergodic and equals the vertical foliation of c .*

In particular, note that as a consequence of Theorem 6.3 and the Masur Criterion, if c, d are two Teichmüller rays in a thick part such that πc and πd fellow travel in $\mathcal{C}(\Sigma)$ then the vertical foliations of c and d determine the same point in \mathcal{PMF} , and this point is $c(\infty) = d(\infty)$.

Lemma 7.5. *Suppose $[X_n, Y_n]$ are Teichmüller geodesic segments that converge to a Teichmüller ray c , so that $X_n \rightarrow c(0)$. If c and all $[X_n, Y_n]$ are in a thick part $\mathcal{T}_{\geq \epsilon}$ then $Y_n \rightarrow c(\infty)$.*

Proof. In the proof we will use Theorem 6.3 as well as Proposition 7.4. First note that if $\pi(Y_n) \rightarrow \pi(c(\infty))$ then $Y_n \rightarrow c(\infty)$ since $c(\infty)$ is uniquely ergodic.

Now suppose that, after a subsequence, $Y_n \rightarrow z \neq c(\infty)$. There are two cases.

Suppose first $\pi(z) \in \partial\mathcal{C}(\Sigma)$. The quasi-geodesics πc and a quasi-geodesic from $\pi(c(0))$ to $\pi(z)$ start at the same point and go to distinct boundary points, so they coarsely form a tripod, with the center point $w \in \mathcal{C}(\Sigma)$, say. Since πc is a (K, L) -quasi-geodesic, we can choose $T \gg 0$ such that any Teichmüller geodesic segment of length T in $\mathcal{T}_{\geq \epsilon}$ projects in $\mathcal{C}(\Sigma)$ with endpoints at distance $\gg d(\pi c(0), w)$. For large n choose $W_n \in [X_n, Y_n]$ at distance T from X_n . Thus $\pi(W_n)$ is in a bounded neighborhood of a quasi-geodesic from w to $\pi(z)$. But $W_n \rightarrow c(T)$ so for large n $\pi(W_n)$ is in a bounded neighborhood of a quasi-geodesic from w to $\pi(c(\infty))$. This is impossible since $\pi(W_n)$ is also far from w .

The other case is that $\pi(z) \in \mathcal{C}(\Sigma)$, and then $\pi(Y_n)$ coarsely rotate around $\pi(z)$. Again choose $W_n \in [X_n, Y_n]$ at a fixed distance from X_n so that $\pi(W_n)$ is much further from $\pi(X_n)$ than $\pi(z)$. Thus $\pi(W_n)$ also coarsely rotate around $\pi(z)$. After a subsequence, no $\pi(W_n)$ is close to a quasi-geodesic between $\pi(z)$ and $\pi c(\infty)$ giving a contradiction as before. \square

Proof of (F1). We will prove a stronger form of (F1): Suppose $[X_n, Y_n]$ are geodesics in $\mathcal{T}_{\geq \epsilon}$, $X_n \rightarrow X$, $d(X_n, Y_n) \rightarrow \infty$ and $d(Y_n, Z_n)$ is uniformly bounded. Then after a subsequence Y_n and Z_n both converge to the same point in \mathcal{PMF} .

Passing to a subsequence, we may assume that $[X_n, Y_n]$ converge to a geodesic ray c with $c(0) = X$, and c is contained in $\mathcal{T}_{\geq \epsilon}$. Likewise we may assume that $[X_n, Z_n]$ converge to a geodesic ray c' with $c'(0) = X$. Since by Lemma 7.3(i) $[X_n, Z_n]$ are contained in a fixed neighborhood of $[X_n, Y_n]$, it follows that c' is in a neighborhood of c , and the projections of c and c' fellow travel and therefore converge to the same point in $\partial\mathcal{C}(\Sigma)$. By unique ergodicity it follows that $c(\infty) = c'(\infty)$ and Lemma 7.5 implies that both Y_n and Z_n converge to this point. \square

Proof of (F2). First, by Theorem 5.3, for a given t there is a neighborhood U_+ of $c(\infty)$ such that $d(c(t), c'(t)) < 1$ whenever $c'(0) = c(0)$ and the vertical foliation of c' is in U_+ . Since for geodesic rays the

vertical foliation is the same as the limiting point, it suffices to prove the following: if c is in $\mathcal{T}_{\geq \epsilon}$ and $d(c(0), c'(0)) \leq \rho$, $c'(\infty) = c(\infty)$ then $d(c(\tau), c'(\tau)) \leq R$ for some R that depends only on ρ . By Lemma 7.3(i), the geodesic from $c'(0)$ to $c(T)$ is in a fixed neighborhood of c so the same is true for the limiting geodesic as $T \rightarrow \infty$. But this limiting geodesic is c' by Lemma 7.5. \square

Remark 7.6. The question when Teichmüller rays are parallel, or asymptotic, is well understood. See [21, 39, 46, 52, 62].

Proof of (F3). Let $x \in T_{K,\rho}(\xi_-, \xi_+)$. Choose a sequence c_n as in the definition. After a subsequence and reparametrization, $c_n \rightarrow c$. Thus c is a Teichmüller geodesic in $\mathcal{T}_{\geq \epsilon}$ that passes within ρ of x . It remains to prove that if c, c' are two geodesics in the thick part and $c(\pm\infty) = c'(\pm\infty)$, then c, c' are in uniform neighborhoods of each other. By Lemma 7.3(ii), uniformity is automatic if we can show that c, c' are parallel, i.e. contained in each other's metric neighborhoods. Consider the geodesics $[c(0), c'(t)]$. From Lemma 7.5 as $t \rightarrow \pm\infty$ they converge to c , implying the claim, since by Lemma 7.3(i) again all these segments are in a fixed neighborhood of c' . \square

Remark 7.7. The Gardiner-Masur theorem [27] states that if ξ_{\pm} are two measured foliations such that $i(\xi_+, \mu) + i(\xi_-, \mu) > 0$ for every $\mu \in \mathcal{MF}$, then there is a unique Teichmüller geodesic with vertical and horizontal foliations ξ_{\pm} . Conversely, any pair ξ_{\pm} of a horizontal and vertical foliation satisfies this condition.

Remark 7.8. We stated the axioms with an eye towards applying them to $\text{Out}(F_n)$. At present we do not know how to make them all work, but we point out the following. For $R \gg 0$ consider biinfinite folding paths that are contained in the $\frac{1}{R}$ -thick part of Culler-Vogtmann's Outer space and whose projections to the complex of free factors are (R, R) -quasi-geodesics. The main ingredients needed for the flow axioms are all known in the $\text{Out}(F_n)$ context: these lines are contracting [13], the Masur Criterion holds [59], and the analog of Klarreich's theorem is in [15, 31]. The projection axioms are more subtle, but see [14].

8. TEICHMÜLLER GEODESICS INTERSECTING THE THIN PART

Suppose $X \in \mathcal{T}_{\geq \epsilon}$ and $\xi \in \mathcal{PMF}$. Section 7 was concerned with Teichmüller rays completely contained in the thick part. Here we want to establish that if there are short curves along the ray, then there is a (proper) subsurface where the projection distance between the vertical foliation and a suitable point thought of as the projection of the initial point of the ray, is large.

The following theorem is an easy consequence of the work of Rafi [60].

Theorem 8.1. *For each collection \mathbf{Y}^i (see Section 6.H) fix a basepoint $X^i \in \mathbf{Y}^i$, and also fix $X_0 \in \mathcal{T}$. Then for every $\Theta > 0$ there is a compact set $K \subset \mathcal{T}$ such that for any $(g, \xi) \in G \times \mathcal{PMF}$ one of the following holds:*

- (a) *The Teichmüller geodesic ray $c_{gX_0, \xi}$ is contained in $G \cdot K$, or*
- (b) *there is some \mathbf{Y}^i and $Y \in \mathbf{Y}^i$ so that $d_Y^\pi(gX^i, \xi) > \Theta$.*

Proof. If ξ is not filling we can take Y to be a component of $\text{Supp}(\xi)$ and then (b) holds since $d_Y^\pi(gX^i, \xi) = \infty$. So assume ξ is filling. By equivariance, we can also assume $g = 1$. Fix a finite collection of curves β_1, \dots, β_m that fill Σ . Note that there is a bound on the intersection numbers between any β_j and any ∂X^i . This means that when $d_Y^\pi(\beta_j, X^i)$ is defined, it is uniformly bounded.

Now suppose that there is a curve s that has length $< \epsilon$ somewhere along $c = c(X_0, \xi)$. Choose $\beta = \beta_j$ that intersects s .

Denote by q^H and $q^V = \xi$ the horizontal and vertical foliations of c . By [60, Theorem 5.6] applied to $c(t)$ there is a component Z of $X - s$ such that

$$\max\{i_Z(\beta, q^H), i_Z(\beta, q^V)\}$$

is as large as we want, provided ϵ is sufficiently small (here i_Z stands for the intersection number between components of intersection of $Z \cap \beta$ and $Z \cap |\xi|$).

Since s is necessarily mostly vertical in X (otherwise it will never get short) it follows from [60, Theorem 5.5] that $i_Z(\beta, q^H)$ is bounded. Thus $i_Z(\beta, q^V)$ is large and by Lemma 6.10 there is a further subsurface Y of Z such that $d_Y^\pi(\beta, \xi)$ is large. Thus Y satisfies the conclusion, since replacing β by X^i (where $Y \in \mathbf{Y}^i$) changes projection distance by a bounded amount.

If no curve gets ϵ -short along c , then (a) holds with a suitable K (see Theorem 5.1). \square

Remark 8.2. Here we summarize the above proof. The idea is that ξ filling guarantees that $i(s, \xi) > 0$ and then cutting along s decomposes ξ into a bounded number of (vertical) rectangles each of which has a very small horizontal measure (since s does). By a version of the Collar Lemma for quadratic differentials that goes back to Minsky [56, Theorem 4.5] some component of $\beta - s$ intersects these rectangles in segments whose total horizontal measure is bounded away from 0. Thus by the pigeon-hole principle β intersects one of the rectangles many times, hence the large intersection number.

9. THE FARRELL-JONES CONJECTURE FOR MAPPING CLASS GROUPS

In this section we prove that mapping class groups satisfy the Farrell-Jones Conjecture. We fix a closed oriented surface Σ of genus g with p punctures and write $\text{Mod}(\Sigma)$ for its mapping class group.

We start with a proof of Theorem B, which we restate for convenience.

Theorem B. *Let Σ be a closed oriented surface of genus g with p punctures. Assume that $6g+2p-6 > 0$. Then the action of its mapping class group $\text{Mod}(\Sigma)$ on the space \mathcal{PMF} of projective measured foliations on Σ is finitely \mathcal{F} -amenable where \mathcal{F} is the family of subgroups that are either virtually cyclic or virtually fix a subsurface of Σ .*

Earlier we described \mathcal{F} as the family of subgroups that virtually fix a curve or are virtually cyclic. This is the same family, because a subgroup that fixes a subsurface fixes all boundary curves of the subsurface virtually, and if a subgroup fixes a curve, then it also fixes the corresponding annulus.

Proof of Theorem B. By Proposition 4.4 it suffices to show that the action of the color preserving subgroup $G < \text{Mod}(\Sigma)$ (see Section 6.H) is finitely \mathcal{F} -amenable. Let $T = \mathcal{T}$ be the Teichmüller space of Σ and let $\Delta = \mathcal{PMF}$. Thus $\overline{T} = T \cup \Delta$ is homeomorphic to a ball and T is its interior. To finish defining the flow data, for a compact set $K \subset T$ let \mathcal{G}_K be the set of Teichmüller rays which are contained in $G \cdot K$. By the Masur Criterion (see Proposition 7.4 every such ray converges to a point of Δ .

We now define the projection data. For a subsurface $Y \subset \Sigma$ let $\Delta(Y) \subset \Delta$ be the open set of foliations whose support is not disjoint from Y . The collection of subsurfaces which are not pairs of pants is partitioned into subcollections $\mathbf{Y}^1, \dots, \mathbf{Y}^k$ each of which is a G -orbit and consists of overlapping subsurfaces (i.e. $Y, Y' \in \mathbf{Y}^i$, $Y \neq Y'$ implies $\partial Y \cap \partial Y' \neq \emptyset$). We set $\mathcal{Y} = \{\mathbf{Y}^1, \dots, \mathbf{Y}^k\}$. Projections and projection distance $d_Y^\pi(X, Z)$ was defined in Sections 6.D and 6.F and $d_Y^\pi(X, \xi)$ for $\xi \in \Delta(Y)$ in Sections 6.E and 6.F. Axiom (P1) is obvious, (P2) was recorded as Proposition 6.4, (P3) is Proposition 6.5, (P4) is Proposition 6.6 and (P5) is Proposition 6.8. This finishes the projection axioms.

Axiom 1.5 holds by Theorem 8.1 and the flow axioms were proved in Section 7. \square

Corollary 9.1. *Assume $6g + 2p - 6 > 0$. Then the action of the Mapping class group $\text{Mod}(\Sigma)$ on the Thurston compactification $\overline{\mathcal{T}} =$*

$\mathcal{T} \cup \mathcal{PMF}$ of Teichmüller space is finitely \mathcal{F} -amenable where \mathcal{F} is the family of subgroups that are either virtually cyclic or stabilize a subsurface.

Proof. By Theorem B the action of $\text{Mod}(\Sigma)$ on \mathcal{PMF} is N - \mathcal{F} -amenable for some N . Let $N' := \dim \mathcal{T}$. We will show that the action of $\text{Mod}(\Sigma)$ on $\overline{\mathcal{T}}$ is $N + N' + 1$ - \mathcal{F} -amenable.

Let $S \subseteq \text{Mod}(\Sigma)$ be finite. Then there exists an open \mathcal{F} -cover of the product $\text{Mod}(\Sigma) \times \mathcal{PMF}$ of order at most N that is S -long in the group coordinate. It is not difficult to extend this cover to a $\text{Mod}(\Sigma)$ -invariant collection \mathcal{U} of open \mathcal{F} -subsets of $\overline{\mathcal{T}}$ of order at most N satisfying

$$\forall (g, \xi) \in \text{Mod}(\Sigma) \times \mathcal{PMF} \exists U \in \mathcal{U} \text{ with } gS \times \{\xi\} \subseteq U,$$

see Lemma 3.16. Teichmüller space \mathcal{T} carries the structure of a proper $\text{Mod}(\Sigma)$ - CW -complex. (Even simpler, we could restrict here to a finite index subgroup of $\text{Mod}(\Sigma)$ that acts freely.) We therefore find an open Fin -cover \mathcal{V} of \mathcal{T} of dimension N , where Fin denotes the family of finite subgroups. Now the cover of $\text{Mod}(\Sigma) \times \mathcal{PMF}$ consisting of all $U \in \mathcal{U}$ and all sets of the form $\text{Mod}(\Sigma) \times V$ with $V \in \mathcal{V}$ is an open \mathcal{F} cover, is of order at most $N + N' + 1$, and is S -long in the group coordinate. \square

We now write $c(\Sigma) := 6g + 2p - 6$. For a subsurface Y of Σ we write Y' for the closed surface obtained by collapsing the boundary curves of Y to punctures. Note that $c(Y') < c(\Sigma)$.

Lemma 9.2. *Let G be a subgroup of $\text{Mod}(\Sigma)$ that fixes a subsurface Y of Σ . Then there exists a finite index subgroup G_0 of G and a central extension*

$$\mathbb{Z}^k \xrightarrow{i} G_0 \xrightarrow{p} Q$$

where Q is a subgroup of a product of mapping class groups of closed surfaces \hat{Y}_i with punctures for which $c(\hat{Y}_i) < c(\Sigma)$.

Proof. Let $Y_0 := Y$ and let Y_1, \dots, Y_n be the components of the complement. For each Y_i that's not an annulus let \hat{Y}_i be the surface obtained from Y_i by collapsing the boundary components to punctures. Then $c(\hat{Y}_i) < c(\Sigma)$. Since G permutes the Y_i 's and their boundary components, there is a finite index subgroup G_0 of G that preserves each Y_i and each boundary component of Y_i . Restriction to each non-annular Y_i followed by capping off the boundary components induces a group homomorphism [23, Proposition 3.19]

$$G_0 \xrightarrow{p} \text{Mod}(\Sigma)_0 \times \dots \times \text{Mod}(\Sigma)_n.$$

whose kernel is the free abelian group generated by Dehn twists around boundary curves of Y and is central in G_0 . \square

We will now use some of the classes of groups introduced in Section 4.B.

Lemma 9.3. *The mapping class group $\text{Mod}(\Sigma)$ of Σ belongs to the class of groups $\text{AC}(\text{VNil})$.*

Proof. We will proceed by induction on $c(\Sigma)$. In the sporadic case $c(\Sigma) = 0$ we have either $g = 0$, $p \leq 3$ or $g = 1$, $p = 0$. If $g = 0$, $p \leq 3$, then $\text{Mod}(\Sigma)$ is finite and belongs to $\text{AC}(\text{VNil})$. If $g = 1$, $p = 0$, then $\text{Mod}(\Sigma) = \text{SL}_2(\mathbb{Z})$ acts cocompactly on a locally finite tree T . The geodesic ray compactification \overline{T} of the tree is a compact ER and the action of $\text{SL}_2(\mathbb{Z})$ on \overline{T} is finitely VCyc-amenable by [6]. Alternatively, we could use the action of $\text{SL}_2(\mathbb{Z})$ on the hyperbolic plane \mathbb{H}^2 as discussed in Example 1.9. Thus $\text{Mod}(\Sigma) \in \text{AC}(\text{VNil})$.

Suppose now $c(\Sigma) > 0$. The Thurston compactification $\overline{\mathcal{T}(\Sigma)}$ of Teichmüller space is homeomorphic to a closed Euclidean ball and in particular a compact ER . Virtually cyclic groups belong to VNil and therefore also to $\text{AC}(\text{VNil})$. Lemma 4.9 (b) implies that $\text{AC}(\text{VNil})$ is closed under finite products, central extensions with finitely generated kernel, taking subgroups and with taking overgroups of finite index. Therefore the induction hypothesis and Lemma 9.2 imply that all stabilizers of subsurfaces of Σ belong to $\text{AC}(\text{VNil})$. Now Corollary 9.1 allows us to use the defining property of the operation \mathbf{ac} . Therefore $\text{Mod}(\Sigma)$ belongs to $\text{AC}(\text{VNil})$. \square

Proof of Theorem A. Corollary 4.10 and Proposition 4.11 imply that all groups in $\text{AC}(\text{VNil})$ satisfy the Farrell-Jones Conjecture. Since mapping class groups of surfaces belong to this class by Lemma 9.3 they satisfy the Farrell-Jones Conjecture. \square

Remark 9.4. The *Farrell Jones Conjecture with wreath products* [9, Sec. 6] is a strengthening of the Farrell Jones Conjecture that has the advantage that it is closed under taking finite index overgroups. Since the class $\text{AC}(\text{VNil})$ is closed under taking finite overgroups and under finite products all groups in $\text{AC}(\text{VNil})$ satisfy the Farrell-Jones Conjecture with wreath products. In particular, mapping class groups of surfaces satisfy the Farrell-Jones Conjecture with wreath products. This is also true for all groups in the class $\text{AC}(\text{VSol})$.

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